

MATINGS WITH LAMINATIONS

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ABSTRACT. We give a topological description of the space of quadratic rational maps with superattractive two-cycles: its “non-escape locus” \mathcal{M}_2 (the analog of the Mandelbrot set \mathcal{M}) is locally connected, it is the continuous image of \mathcal{M} under a canonical map, and it can be described as \mathcal{M} (minus the $1/2$ -limb), mated with the lamination of the basilica. The latter statement is a refined version of a conjecture of Ben Wittner, which in its original version requires local connectivity of \mathcal{M} to even be stated. Our methods of mating with a lamination also apply to dynamical matings of certain non-locally connected Julia sets.

1. INTRODUCTION

A convenient way to describe some rational maps is by means of matings. If a rational map is a mating of two polynomials, then the dynamics of this rational map is a sum of polynomial dynamics. In addition, dynamical matings usually, if not always, have consequences to the parameter spaces. A classical example is the space of quadratic rational functions with superattractive n -cycles; and, in particular, the case $n = 2$ which will be the main object of our study.

In this paper we introduce a surgery “matings with laminations” that generalizes a classical mating construction for certain cases when one of the sets is not locally connected (some infinitely renormalizable quadratic polynomials) or local connectivity is an open question (the Mandelbrot set). Our results could be easily generalized, for example, to certain components of the space of quadratic rational functions with superattractive n -cycles.

1.1. Wittner’s conjecture. Consider the space \mathcal{V}_1 of quadratic polynomials parametrized by $c \in \mathbb{C}$ via

$$\{f_c(z) = z^2 + c : c \in \mathbb{C}\}.$$

Many important dynamical properties of \mathcal{V}_1 are encoded in the *Mandelbrot set*

(1)

$\mathcal{M} = \{c \in \mathbb{C} : \text{the polynomial } f_c : z \rightarrow z^2 + c \text{ has connected Julia set}\}.$

For instance, the boundary of the Mandelbrot set is the bifurcation locus: $z^2 + c$ is structurally stable if and only if $c \notin \partial\mathcal{M}$.

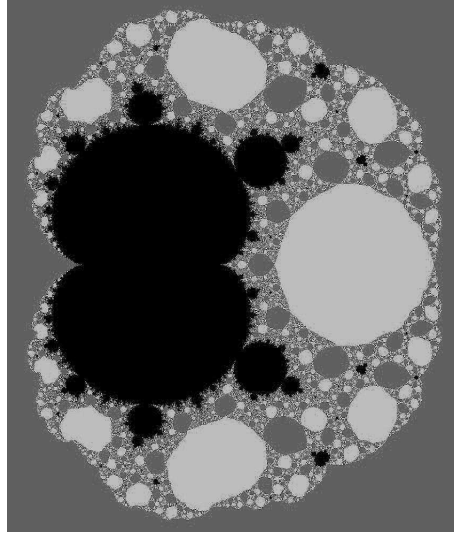


FIGURE 1. The parameter slice \mathcal{V}_2 of quadratic rational functions with superattractive two cycles parametrized as $\{\frac{a}{z^2+2} : a \in \mathbb{C} \setminus \{0\}\}$. The set \mathcal{M}_2 (the black set on the figure) is the set of parameters such that the second critical point is not in the attracting basin of the two-cycle. Figure from [Ti2].

McMullen’s motto “the Mandelbrot set is universal” [Mc2] states that there are infinitely many copies of the Mandelbrot set in every non-trivial one dimensional parameter space of rational functions with one “active” (and non-degenerate) critical point. The Douady-Hubbard straightening theorem [DH1] gives a canonical dynamically meaningful homeomorphism between the copies and \mathcal{M} . In particular, there are infinitely many copies of \mathcal{M} within the Mandelbrot set itself, even within every open set intersecting $\partial\mathcal{M}$.

The Mandelbrot set may occur in more ways than predicted by the straightening theorem. Following the notation of Mary Rees, denote by \mathcal{V}_2 the space (slice) of quadratic rational maps that have superattractive two-cycles and by $\mathcal{M}_2 \subset \mathcal{V}_2$ the set of non-escaping parameters (see Figure 1 and its caption). As in \mathcal{V}_1 , a map $g \in \mathcal{V}_2$ is structurally stable if and only if $g \notin \partial\mathcal{M}_2$. Further, denote by K_B the filled-in Julia set of the Basilica polynomial $f_B(z) = z^2 - 1$. Maps in \mathcal{M}_2 should be thought as matings of polynomials in \mathcal{M} with f_B ; this has been already worked out in many cases [Re], [Tan], [Lu], [AY], [Ti1] but, in general, is not correct as there are quadratic polynomials with non locally connected Julia sets. In 1988 Wittner conjectured [Wi] a parameter counterpart to the dynamical matings:

$$(2) \quad \mathcal{V}_2 \text{ is the mating } \mathcal{M} \cup_{\mathbb{S}^1} K_B.$$

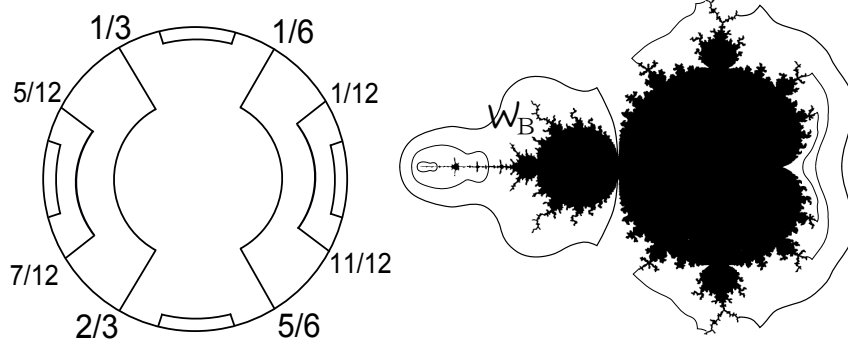


FIGURE 2. Left: the lamination L_B of the Basilica. The $\langle 1/3, 2/3 \rangle$ and $\langle 5/6, 1/6 \rangle$ leaves are symmetric; any other leaf is an iterated pre-image of the $\langle 5/6, 1/6 \rangle$ leaf under z^2 . Right: the embedding \mathcal{L}_B of L_B into \mathcal{V}_1 ; the *Basilica domain* \mathcal{W}_B is a closed disc surrounded by the $\langle 1/3, 2/3 \rangle$ -leaf and containing the Basilica limb.

More precisely, remove the $1/2$ -limb (the Basilica limb) from the Mandelbrot set, remove the $1/2$ -limb from K_B , and glue these sets along the boundaries parametrized by parameter and dynamic rays; this construction requires $\partial\mathcal{M}$ (and ∂K_B) to be locally connected. After the mating, hyperbolic components of the Mandelbrot set become hyperbolic components of mating type, Fatou components of K_B become hyperbolic components of capture type, and the boundary of the Mandelbrot set (or, equivalently, ∂K_B) becomes the bifurcation locus $\partial\mathcal{M}_2$ of \mathcal{V}_2 .

1.2. Main results. We give here a refinement of Wittner's construction that is unconditional to the local connectivity of $\partial\mathcal{M}$. Briefly, we write $K_B = \overline{\mathbb{D}}/L_B$, see the next paragraph, so $\mathcal{M} \cup_{\mathbb{S}^1} K_B = \mathcal{M} \cup_{\mathbb{S}^1} (\overline{\mathbb{D}}/L_B)$. Then we change the order of operations:

$$\mathcal{M} \cup_{\mathbb{S}^1} L_B := (\mathcal{M} \cup_{\mathbb{S}^1} \overline{\mathbb{D}})/L_B = \mathcal{V}_1/L_B,$$

where $\mathcal{M} \cup_{\mathbb{S}^1} \overline{\mathbb{D}} = \mathcal{V}_1$ is a trivial mating (just extending \mathcal{M} into $\hat{\mathbb{C}}$). We call $\mathcal{M} \cup_{\mathbb{S}^1} L_B$ the *mating with the lamination*. If $\partial\mathcal{M}$ is locally connected, then $\mathcal{M} \cup_{\mathbb{S}^1} L_B$ is the same as $\mathcal{M} \cup_{\mathbb{S}^1} K_B$.

Let us explain the construction in more detail. Consider the *lamination* L_B of the Basilica (Figure 2); L_B is a family of disjoint curves, called *leaves*, in the closed unit disc $\overline{\mathbb{D}}$ connecting points on $\partial\overline{\mathbb{D}}$ such that L_B is closed as a subset of $\overline{\mathbb{D}}$ and the equivalence relation induced by L_B

$$x \sim_{L_B} y \text{ if and only if } x, y \text{ are in a leaf of } L_B \text{ or } x = y$$

models K_B ; i.e. $K_B = \overline{\mathbb{D}}/L_B$.

Recall that the parameter Böttcher function \mathcal{B} maps $\mathbb{C} \setminus \mathcal{M}$ conformally to $\mathbb{C} \setminus \overline{\mathbb{D}}$. Denote by

$$\mathcal{L}_B = \{\mathcal{B}^{-1} \circ (1/z)(l) : l \in L_B\}$$

the embedding of L_B into $\mathcal{V}_1 \setminus \mathcal{M}$; let us remark that $\mathcal{B}^{-1} \circ (1/z)(l)$ is well defined and is a closed curve as the ends of $\mathcal{B}^{-1} \circ (1/z)(l)$ land at Misiurewicz or parabolic parameters at which \mathcal{M} has triviality of the fiber.

We say that two non-equal points are equivalent under \mathcal{L}_B if these points are either in the closure of a leaf in \mathcal{L}_B or in the Basilica domain \mathcal{W}_B (see Figure 2 and its caption) which is a closed neighborhood of the Basilica limb.

Define \mathcal{V}'_2 to be the quotient $\mathcal{V}_1/\mathcal{L}_B$, define $m : \mathcal{V}_1 \rightarrow \mathcal{V}_1/\mathcal{L}_B = \mathcal{V}'_2$ to be the quotient map, and define $\mathcal{M}'_2 := m(\mathcal{M})$. A priori, the construction of \mathcal{V}'_2 depends on the choice of leaves in L_B .

Theorem 1.1 (The mating $\mathcal{M} \cup_{\mathbb{S}^1} L_B$ is well defined). *The restriction $m|_{\mathcal{M}} : \mathcal{M} \rightarrow m(\mathcal{M}) = \mathcal{M}'_2$ is canonical (independent of L_B). The space \mathcal{M}'_2 is locally connected. All spaces $\mathcal{V}'_2 = \mathcal{V}'_2(L_B)$ for different choices of leaves in L_B are homeomorphic by homeomorphisms preserving \mathcal{M}'_2 .*

The next theorem says that $\mathcal{M} \cup_{\mathbb{S}^1} L_B$ is topologically and combinatorially isomorphic to \mathcal{V}_2 .

Theorem 1.2 (Refined Wittner's Conjecture). *The space $\mathcal{M} \cup_{\mathbb{S}^1} L_B$ is homeomorphic to \mathcal{V}_2 by a homeomorphism that preserves the combinatorics of parapuzzle pieces and coincides with straightening maps on all small copies of the Mandelbrot set. This homeomorphism is canonical (unique) over \mathcal{M}_2 .*

Corollary 1.3. *The set \mathcal{M}_2 is locally connected and is a continuous image of \mathcal{M} .*

By Theorem 1.2 we have $\mathcal{M}'_2 \approx \mathcal{M}_2$ (canonically) and we view the map $m|_{\mathcal{M}}$ in Theorem 1.1 as a map from \mathcal{M} to \mathcal{M}_2 . In fact, $m|_{\mathcal{M}}$ has a simple description: it collapses the basilica limb to a point and pinches (identifies) Misiurewicz parameters with external angles ϕ_1, ϕ_2 if and only if ϕ_1, ϕ_2 are identified by L_B .

Similarly to $\mathcal{M} \cup_{\mathbb{S}^1} L_B$ we define the dynamical mating $f_c \cup_{\mathbb{S}^1} L_B$, see details in Section 7. The last theorem is the dynamical counterpart to Theorem 1.2.

Theorem 1.4 ($f_c \cup_{\mathbb{S}^1} L_B \approx g_{m(c)}$). *If $c \in \mathcal{M} \setminus \mathcal{W}_B$ is not on the boundary of a hyperbolic component or c is at least 4 times renormalizable, then $f_c \cup_{\mathbb{S}^1} L_B$ is well defined and is topologically conjugate to $g_{m(c)}$.*

If the Julia set of f_c is locally connected, then

$$f_c \cup_{\mathbb{S}^1} L_B = f_c \cup_{\mathbb{S}^1} f_B.$$

This gives a topological description of $g_{m(c)}$ modulo f_c . For instance, $f_c : J_c \rightarrow J_c$ is semi-conjugate to $g_{m(c)} : J_{m(c)} \rightarrow J_{m(c)}$; the semi-conjugacy is unique if it preserves the combinatorics of puzzle pieces and coincides with straightening maps on small filled in Julia sets.

It has been a long standing problem to describe maps in \mathcal{V}_2 in terms of surgeries over quadratic polynomials. Hyperbolic maps in \mathcal{M}_2 (resp. in $\mathcal{V}_2 \setminus \mathcal{M}_2$) are matings (resp. captures) with the Basilica polynomial [Re], [Tan], [Lu]; this result is an application of Thurston's topological characterization of postcritically finite rational functions [DH3]. Maps on the external boundary of \mathcal{M}_2 are described in [Ti1] in terms of anti-matings. In [AY] existence and uniqueness of the matings of the Basilica with non-renormalizable Yoccoz's polynomials in $\mathcal{M} \setminus \mathcal{W}_B$ are shown; this result could be easily generalized for the finite-renormalizable case. Theorem 1.4 contributes the infinitely renormalizable case. Because J_c can be non-locally connected the mating $f_c \cup_{\mathbb{S}^1} L_B$ in Theorem 1.4 can not be always replaced by $f_c \cup_{\mathbb{S}^1} f_B$.

1.3. Parameter slices \mathcal{V}_n . Similarly to \mathcal{V}_1 and \mathcal{V}_2 , denote by \mathcal{V}_n the space (slice) of quadratic rational maps that have superattractive n -cycles and by $\mathcal{M}_n \subset \mathcal{V}_n$ the set of non-escaping parameters; again, $\partial\mathcal{M}_n$ is the bifurcation locus.

For $n \geq 3$ the slice \mathcal{V}_n consists of finitely many components; some of these components have similar descriptions as \mathcal{V}_2 . For example, \mathcal{V}_3 (Figure 3) consists of the Rabbit part \mathcal{V}_R , the Corabbit part \mathcal{V}_C , the Airplane part \mathcal{V}_A , and two main hyperbolic capture components. The proofs of Theorem 1.1, 1.2, 1.4 work for \mathcal{V}_R and \mathcal{V}_C with slight modifications so \mathcal{V}_R and \mathcal{V}_C are the matings of the Mandelbrot set with the laminations of the Rabbit and Corabbit respectively. In general, for an n -rabbit $f_{R'}$ (i.e. $f_{R'}$ is the center of a hyperbolic component attached to the main hyperbolic component of \mathcal{M}), define $\mathcal{V}_{R'}$ to be the part of \mathcal{V}_n consisting of “matings and captures with $f_{R'}$ ”. Then $\mathcal{V}_{R'}$ is $\mathcal{M} \cup_{\mathbb{S}^1} L_{R'}$.

1.4. Outline of the paper and conventions. In Section 2 (Background: \mathcal{V}_1) we review properties of \mathcal{V}_1 .

A puzzle piece (Subsection 2.2) is a closed topological disc in dynamical plane bounded by (finitely many) periodic and pre-periodic dynamic rays and truncated by an equipotential such that the forward orbit of the boundary of a puzzle piece does not intersect its interior.

Using Böttcher functions we combinatorially identify puzzle pieces in different dynamical planes: puzzle pieces $Y(c_1), Y(c_2)$ in the dynamical planes of f_{c_1} and f_{c_2} are equivalent (simply, $Y(c_1) = Y(c_2) = Y$) if the composition $B_{c_2}^{-1} \circ B_{c_1}$ of Böttcher functions induces a homeomorphism between $\partial Y(c_1) \setminus J_{c_1}$ and $\partial Y(c_2) \setminus J_{c_2}$ that extends to a homeomorphism between $\partial Y(c_1)$ and $\partial Y(c_2)$. Equivalently, there is a homeomorphism

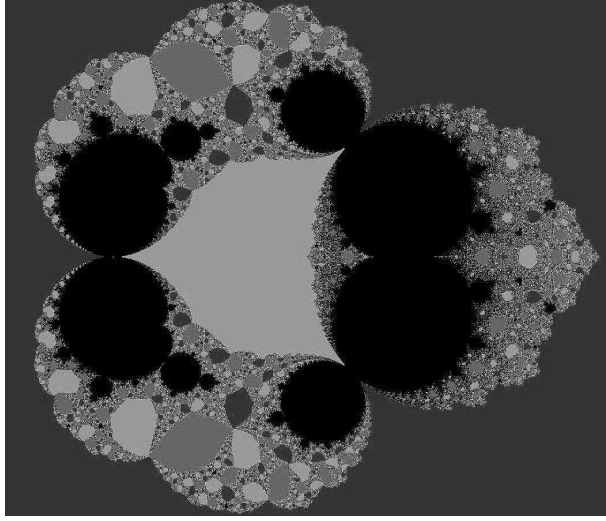


FIGURE 3. The parameter slice \mathcal{V}_3 consists of \mathcal{V}_R , \mathcal{V}_C (symmetric to \mathcal{V}_R), and two main hyperbolic capture component. Figure from [Ti2].

between $\partial Y(c_1)$ and $\partial Y(c_2)$ preserving angles and heights of rays and equipotentials.

A closed disc \mathcal{Y} in the parameter plane bounded by parameter rays and truncated by a parameter equipotential is a parapuzzle piece if \mathcal{Y} is the parameter counterpart to a dynamic puzzle piece, denoted by Y : the composition $\mathcal{B}^{-1} \circ B_c$ of Böttcher functions induces a homeomorphism between $\partial Y(c) \setminus J_c$ and $\partial \mathcal{Y} \setminus \mathcal{M}$ as above.

We will use the following convention: objects in the parameter plane will be denoted by calligraphic capital letters (such as \mathcal{Z}_i^n , \mathcal{R}^ϕ) while those in dynamical planes will be denoted by Roman capitals (such as Z_i^n , R^ϕ).

In Subsection 2.4 we recall the construction of straightening maps; small copies of the Mandelbrot set are discussed Subsection 2.5. In Subsection 2.6 we review the decoration theorem. In Subsection 2.7 we discuss compact connected locally connected subsets of the 2-sphere. In Subsection 2.8 we fix some conventions regarding laminations.

In Section 3 (Mating construction) we will prove Proposition 3.8 which is a refined version of Theorem 1.1. In Subsection 3.1 we discuss the relation between small copies of the Mandelbrot set in \mathcal{M} and leaves in \mathcal{L}_B . We will show that points on the boundary of a component U of $\mathcal{V}_1 \setminus (\mathcal{M} \cup \mathcal{L}_B \cup \mathcal{W}_B)$ are not infinitely renormalizable; this will imply that ∂U is locally connected (the first part of Proposition 3.8).

Further, using the decoration theorem (Subsection 3.2) we will show that *big* leaves (with diameters at least a fixed $\varepsilon > 0$) of \mathcal{L}_B and big components of $\mathcal{V}_1 \setminus (\mathcal{M} \cup \mathcal{L}_B \cup \mathcal{W}_B)$ can not accumulate at secondary copies of the Mandelbrot set (Propositions 3.2, 3.3, and 3.4). Combined

with Yoccoz's results this, on one hand, shows that \mathcal{L}_B is a closed equivalence relation in \mathcal{V}_1 (and canonical on \mathcal{M}), and, on the other hand, leads to local connectivity of $\mathcal{M} \cup \mathcal{L}'_B \cup \mathcal{W}_B$.

Finally, we establish local connectivity of \mathcal{V}'_2 and show that \mathcal{V}'_2 does depend (up to homeomorphism preserving combinatorics) on the choice of leaves in L_B .

In Section 4 we review properties of \mathcal{V}_2 . We parameterize \mathcal{V}_2 as

$$\{g_a(z) = a/(z^2 + 2z) : a \in \mathbb{C} \setminus \{0\}\} \cup \{g_\infty = 1/z^2\},$$

so that $-1, \infty$ are the critical points of g_a and $g_a(0) = \infty$, $g_a(\infty) = 0$.

In Subsection 4.2 we define *rays in bubbles* for g_a ; using these rays we define bubble puzzle pieces. There is a natural identification of rays in bubbles for different dynamical planes; this allows us to combinatorially identify bubble puzzle pieces in different planes. Bubble parapuzzle pieces are “the parameter counterparts” of dynamic bubble pieces.

A special map in \mathcal{V}_2 is $g_1 = 1/(z^2 + 2)$; this map also belongs to \mathcal{V}_1 : it is conjugate by $-1/(z + 1)$ to f_B . Using the dynamical plane of $g_1 \sim f_B$ we define “ \mathcal{V}_2 -twins” (Definition 4.5 and Subsection 4.5) for many structures in \mathcal{V}_1 such as rays, puzzle and parapuzzle pieces, small copies of the Mandelbrot set.

We will often use an upper index B (for example: $Y_{\mathcal{H}}^B$, $\mathcal{Y}_{n\mathcal{H}}^B$, and $\mathcal{M}_{\mathcal{H}}^B$) to indicate the \mathcal{V}_2 -twins of objects in \mathcal{V}_1 (respectively: $Y_{\mathcal{H}}^P$, $\mathcal{Y}_{n\mathcal{H}}^{np}$, and $\mathcal{M}_{\mathcal{H}}$).

In Section 5 (Landing of rays in \mathcal{V}_2) we will prove Theorem 5.1. This will imply that for a parapuzzle piece \mathcal{Y} in $\mathcal{V}_1 \setminus \mathcal{W}_B$ there is its \mathcal{V}_2 -twin \mathcal{Y}^B and every bubble parapuzzle piece is a \mathcal{V}_2 -twin of a parapuzzle piece in $\mathcal{V}_1 \setminus \mathcal{W}_B$. Moreover, a dynamic puzzle piece X exists in $\text{int}(\mathcal{Y})$ if and only if the \mathcal{V}_2 -twin X^B exists in \mathcal{Y}^B . In other words, the parapuzzle partition in $\mathcal{V}_1 \setminus \mathcal{W}_B$ is combinatorially the same as the bubble parapuzzle partition in \mathcal{V}_2 ; this reflects the fact that hyperbolic maps in \mathcal{M}_2 are the mating of hyperbolic polynomials in $\mathcal{M} \setminus \mathcal{W}_B$ with the Basilica polynomial.

In Section 6 we prove Theorem 1.2. By a *combinatorial disc* in \mathcal{V}_2 we mean a closed topological disc \mathcal{D} such that $\partial\mathcal{D}$ is in a finite union of parameter bubble rays with rational angles and simple arcs in closures of hyperbolic components. A sequence a_n *tends to a_∞ combinatorially* if for every combinatorial disc \mathcal{D} such that $a_\infty \in \text{int}(\mathcal{D})$ all but finitely many a_n are in \mathcal{D} . Similarly, combinatorial convergence in dynamical planes is defined.

As in \mathcal{V}_1 , a parameter $a \in \mathcal{V}_2$ is *non-infinitely renormalizable* if a belongs to at most finitely many copies of \mathcal{M} . Theorem 6.8 (Yoccoz's results in \mathcal{V}_2) shows that for a non-infinitely renormalizable parameter a a sequence a_n tends to a in the combinatorial topology if and only if a_n tends to a in the usual topology. In Subsection 6.1 we establish

the decoration theorem for \mathcal{V}_2 (Proposition 6.5) using the straightening theorem.

An isomorphism between \mathcal{V}_2 and \mathcal{V}'_2 is constructed in Subsection 6.5.

In Section 7 we will prove Theorem 1.4.

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2. BACKGROUND: \mathcal{V}_1

2.1. Böttcher functions, rays, and equipotentials. For a polynomial $f_c(z) = z^2 + c$ let

$$(3) \quad \rho_c = \lim_{n \rightarrow \infty} \frac{\log |f_c^n(0)|}{2^n}$$

be the escape rate of the critical point 0. There is a unique holomorphic Böttcher function $B_c(z)$ such that

- $B_c(z)$ is a conformal conjugation between f_c in a neighborhood of infinity with the dynamics of z^2 in the disc $\{z \mid \log |z| > \rho_c\}$; and
- $B_c(z)/z \rightarrow 1$ when $z \rightarrow \infty$.

If $c \in \mathcal{M}$, then the Böttcher function maps the basin of attraction of infinity conformally to the exterior of the unit disc; if $c \notin \mathcal{M}$, then 0 and all its pre-images are critical points for the Böttcher function.

The parameter Böttcher function is defined as

$$(4) \quad \mathcal{B}(c) = B_c(c).$$

Theorem 2.1 ([DH2]). *The map $\mathcal{B}(c)$ is a conformal isomorphism from $\mathbb{C} \setminus \mathcal{M}$ to the exterior of the unit disc.*

In particular, it follows from Theorem 2.1 that \mathcal{M} is connected.

Dynamic (resp. parameter) *rays and equipotentials* are defined as pre-images under $B_c(z)$ (resp. under $\mathcal{B}(c)$) of straight rays and concentric circles. Rays and equipotentials are parametrized by *angles* and *heights* respectively.

Consider a dynamic ray $R^\phi = R^\phi(t), t > \rho_c \geq 0$. If the ray does not hit any critical point of the Böttcher function $B_c(z)$ (i.e. either 0 or its pre-images), then $B_c(z)$ is analytically extendable along the ray $R^\phi(t)$ towards the Julia set; this also extends the ray $R^\phi(t)$ to all $t \in (0, \rho_c]$. Further, if the limit $\lim_{t \rightarrow 0} R^\phi(t) = p$ exists, then it is said that R^ϕ *lands at p*. Similarly, landing of parameter rays is defined. It is known that parameter rays and (extendable) dynamic rays with rational (i.e. periodic or pre-periodic) angles *always land* (see for example [Mi1]). For our convenience, we say that a ray does not land if it goes through a critical point.

2.2. Puzzle and parapuzzle pieces. The combinatorics of dynamic and parameter rays are closely related; the cornerstone of this connection is expressed by the following classical result:

Theorem 2.2 (Correspondence Theorem). *Assume that in the dynamical plane of $f_c = z^2 + c$ rays R^{ϕ_1}, R^{ϕ_2} with rational angles land together, separate the critical value c from the critical point 0 ; and assume that no forward iterate of R^{ϕ_1}, R^{ϕ_2} under f_c separate the critical value c from the pair R^{ϕ_1}, R^{ϕ_2} . Then the parameter rays $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ land together and separate c from 0 .*

In the inverse direction: if parameter rays $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ with rational angles land together, then for every c from the interior of the sector $\mathcal{W}(\phi_1, \phi_2)$ bounded by $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ and not containing 0 the rays R^{ϕ_1}, R^{ϕ_2} land together in the dynamical plane of f_c and have the above property. If ϕ_1, ϕ_2 are periodic angles (i.e. they are of the form $m/(2n-1)$) and $c \notin \mathcal{W}(\phi_1, \phi_2)$, then the rays R^{ϕ_1}, R^{ϕ_2} do not land together in the dynamical plane of f_c .

A ray pair R^{ϕ_1}, R^{ϕ_2} satisfying the condition of the above theorem is called *characteristic* [Mil]; the combinatorics of dynamic and parameter rays will be described in full generality in [Sc2].

A *puzzle piece* $X(c)$ in the dynamical plane of $z^2 + c$ is a closed topological disc bounded by periodic and pre-periodic rays and truncated by an equipotential such that the forward orbit of $\partial X(c)$ does not intersect $\text{int}(X(c))$. By construction, the boundary of $X(c)$ consists of finitely many segments which belong to rays or an equipotential. We will use *floor brackets* to untruncate $X(c)$; i.e. $\lfloor X(c) \rfloor$ is the closed set containing $X(c)$ and bounded by the rays that form the boundary of $X(c)$.

Let $X(c_1)$ be a puzzle piece in the dynamical plane of $z^2 + c_1$. We will say that $X(c_1)$ *exists* for $c_2 \neq c_1$ if there exists a puzzle piece $X(c_2)$ in the dynamical plane of $z^2 + c_2$ such that the boundaries of $X(c_1)$ and $X(c_2)$ are *combinatorially equivalent* in the following sense: there is an analytic continuation of the composition $B_{c_2}^{-1} \circ B_{c_1}$ of the Böttcher functions from a neighborhood of infinity (where the composition is defined canonically) to a neighborhood of

$$(\partial \lfloor X(c_1) \rfloor \cup \partial X(c_1)) \setminus \{\text{Julia set of } z^2 + c_1\}$$

such that $B_{c_2}^{-1} \circ B_{c_1}$ maps $\partial X(c_1)$ homeomorphically onto $\partial X(c_2)$ except finitely many points in the Julia sets that are landing points of periodic and pre-periodic rays. Equivalently, there is a homeomorphism between the boundaries $\partial X(c_1)$ and $\partial X(c_2)$ preserving angles and heights of rays and equipotentials. To simplify, we will often write X for $X(c_1)$ or $X(c_2)$; this convention allows us to use the notion of puzzle pieces without referring to a particular dynamical plane.

A *parapuzzle piece* \mathcal{X} is a closed topological disc in the parameter plane bounded by parameter rays and an equipotential such that \mathcal{X}

is the parameter counterpart of a dynamic puzzle piece, denoted by X in the following sense: the composition $\mathcal{B}^{-1} \circ B_c$ of the Böttcher functions extends from a neighborhood of infinity to a neighborhood of $(\partial X \cup \partial[X]) \setminus J_c$ and maps ∂X homeomorphically onto $\partial \mathcal{X}'$ except finitely many points as above. We will write $\mathcal{X} = \mathcal{X}'$ if \mathcal{X}' is the parameter counterpart to X .

The following property is classical (it follows from Theorem 2.2).

Property 2.3. *If the interior of a puzzle piece X_n contains the critical value, then the parameter counterpart \mathcal{X}_n exists. Moreover, the puzzle piece X_n exists in $\text{int}(\mathcal{X}_n)$.*

2.3. Generalized puzzle and parapuzzle pieces. In this section we define generalized puzzle and parapuzzle pieces; these pieces are often appear as the images of puzzle and parapuzzle pieces in other slices under straightening maps; see the right part of Figure 6 for an example.

We say that a closed topological disc $X(c)$ in the dynamical plane of $z^2 + c$ is a *generalized puzzle piece* if:

- $\partial X(c)$ is a Jordan curve such that $\partial X(c) \cap K_c$ consists of finitely many periodic and preperiodic points;
- there is a connected graph $\Gamma = \Gamma(X(c))$ in $\widehat{\mathbb{C}} \setminus K_c$, to which we will refer as *marking*, such that for every component β of $\partial X(c) \setminus K_c$ there is a simple curve γ in $\Gamma \setminus \{\infty\}$ connecting β and infinity;
- the forward iteration of $\partial X(c) \cup \Gamma(X(c))$ does not intersect $\text{int}(X(c))$.

Let us emphasize that the exact choice of markings will be irrelevant for us. We need markings only to distinguish generalized puzzle pieces in the dynamical planes of polynomials with non-connected Julia sets. We will usually omit the reference to markings.

A *pre-image* $X^2(c)$ of a generalized puzzle piece $X(c)$ is the closure of a connected component of $f_c^{-1}(\text{int}(X(c)))$. We define the marking $\Gamma(X^2(c))$ of $X^2(c)$ to be the closure of the union of all components in $f_c^{-1}(\Gamma(X(c))) \setminus \{\infty\}$ intersecting $\partial X^2(c)$. By definition, a pre-image of a generalized puzzle piece is a generalized puzzle piece.

Two generalized puzzle pieces $X(c_1), X(c_2)$ are *combinatorially equivalent* if there is an analytic continuation of $B_{c_2}^{-1} \circ B_{c_1}$ from a neighborhood of infinity (where the composition is defined canonically) to a neighborhood of

$$\partial X(c_1) \cup \Gamma(X(c_1)) \setminus K_c$$

such that $B_{c_2}^{-1} \circ B_{c_1}$ maps $\partial X(c_1) \cup \Gamma(X(c_1)) \setminus K_{c_1}$ homeomorphically onto $\partial X(c_2) \cup \Gamma(X(c_2)) \setminus K_{c_2}$ and this map extends to a homeomorphism between $\partial X(c_1) \cup \Gamma(X(c_1))$ and $\partial X(c_2) \cup \Gamma(X(c_2))$.

A closed topological disc \mathcal{X} is a *generalized parapuzzle piece* if \mathcal{X} is the parameter counterpart to a dynamic generalized puzzle piece

$X = X(c_1)$: the composition $\mathcal{B}^{-1} \circ B_{c_1}$ extends from a neighborhood of infinity to a neighborhood of $\partial X(c_1) \cup \Gamma(X(c_1)) \setminus K_c$ and maps $\partial X(c_1)$ homeomorphically onto $\partial \mathcal{X}$ except at finitely many points as above.

Similar to Property 2.3 we have:

Property 2.4. *If in the dynamical plane of f_{c_0} the critical value c_0 is in the interior of a generalized puzzle piece X , then X has the parameter counterpart \mathcal{X} and X exists for all $c \in \text{int}(\mathcal{X})$.*

Proof. The argument is classical, so we will give a sketch of the proof.

Define

$$\mathcal{X}^o = \{c \in \mathcal{V}_1 : X \text{ exists for } f_c \text{ and } c \in \text{int}(X(c))\}.$$

By assumption, $c_0 \in \mathcal{X}^o$. If $c' \in \mathcal{X}^o$, then for c in a small neighborhood of c' the set $\cup_{n \geq 0} f_c^{-n}(c)$ is disjoint from $\partial X(c)$ because the forward orbit of $\partial X(c)$ does not intersect $\text{int}(X) \ni c$. This implies that \mathcal{X}^o is open.

It follows that $c \in \partial \mathcal{X}^o$ if and only if $\partial X(c)$ hits c or there is a parabolic point in $\partial X \cap K_c$. It is easy to conclude that $\mathcal{X} = \overline{\mathcal{X}^o}$ is the parameter counterpart of X . \square

We say that a generalized puzzle piece X *behaves periodically around the Julia set*, if there is a neighborhood U of K_c such that every connected curve γ in $\partial X \cap U$ is eventually periodic:

$$f_c^{n+p}(\gamma) \cap U \subset f_c^n(\gamma) \cap U$$

for some $n \geq 0$ and $p > 0$. It is clear that this definition is independent on c .

2.4. Analytic families of quadratic-like mappings. Let us recall the construction of the canonical homeomorphisms between copies of the Mandelbrot set [DH1].

A branched covering $f_\lambda : U'_\lambda \rightarrow U_\lambda$ of degree 2 is *quadratic-like* if U'_λ, U_λ are open subsets of \mathbb{C} and the closure of U'_λ is in U_λ .

Let

$$\mathbf{f} = (f_\lambda : U'_\lambda \rightarrow U_\lambda), \quad \lambda \in \Lambda$$

be a family of quadratic-like maps parametrized by a complex analytic 1-manifold Λ .

Define $\mathbf{U} = \{(\lambda, z) : z \in U_\lambda\}$, $\mathbf{U}' = \{(\lambda, z) : z \in U'_\lambda\}$. We assume that \mathbf{U}, \mathbf{U}' are homeomorphic over Λ to $\Lambda \times \overline{\mathbb{D}}$. Denote by $\mathcal{M}_\mathbf{f}$ the set of parameters in Λ such that $f_\lambda : U'_\lambda \rightarrow U_\lambda$ has a connected Julia set.

Let us outline that in the below construction the straightening map $\chi : \Lambda \rightarrow \mathcal{V}_1$ will be canonical over $\mathcal{M}_\mathbf{f}$ (independent of the choice of a tubing T) and will depend on T over $\Lambda \setminus \mathcal{M}_\mathbf{f}$.

A *tubing* is a diffeomorphic embedding T over Λ of the family of annuli

$$Q_R \times \Lambda = \{z : R^{1/2} \leq |z| \leq R\} \times \Lambda$$

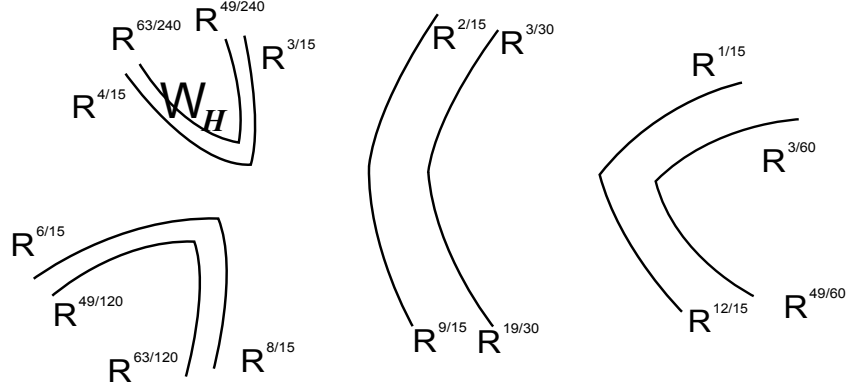


FIGURE 4. Ray configuration in the dynamical plane of $z^2 + c$ for $c \in \mathcal{W}_{\mathcal{H}}$ and \mathcal{H} is the hyperbolic component with external angles $\{3/15, 4/15\}$.

into \mathbf{U} such that $T_{\lambda}(Q_R)$ surrounds the filled-in Julia set of f_{λ} and

$$f_{\lambda}(T_{\lambda}(z)) = T_{\lambda}(z^2) \quad \text{for } |z| = R^{1/2}.$$

Fixing T we get a continuous map

$$\chi = \chi(T) : \Lambda \rightarrow \mathcal{V}_1, \quad \chi(f_{\lambda}) = \{z \rightarrow z^2 + \chi(\lambda)\}$$

[DH1, Theorem 2]. In dynamical planes we have quasiconformal conjugacies, which we also denote by $\{\chi(\lambda)\}_{\lambda}$, between $\{f_{\lambda}\}_{\lambda}$ restricted to the closed discs bounded by the outer boundaries of $\{T_{\lambda}(Q_R)\}_{\lambda}$ and $\{z \rightarrow z^2 + \chi(\lambda)\}_{\lambda}$ restricted to neighborhoods of the filled-in Julia sets. Moreover, the tubing T commutes with the Böttcher coordinates in the following sense:

$$(5) \quad \chi(\lambda_2) \circ T_{\lambda_2} \circ T_{\lambda_1}^{-1} = B_{\chi(\lambda_2)}^{-1} \circ B_{\chi(\lambda_1)} \circ \chi(\lambda_1) \quad \text{on } Q_R.$$

[DH1, Theorem 4] states that if χ is not constant, then it is topologically holomorphic over \mathcal{M} (for instance, χ is locally a homeomorphism over \mathcal{M} except a discrete set); for some T the map χ is topologically holomorphic on a neighborhood of $\mathcal{M}_{\mathbf{f}}$ (see examples in [Ly2]). If, in addition, the degree of χ on $\mathcal{M}_{\mathbf{f}}$ is 1, then χ is a homeomorphism over $\mathcal{M}_{\mathbf{f}}$.

Let us remark that using the puzzle technique we often get a family $\mathbf{f} = (f_{\lambda} : U'_{\lambda} \rightarrow U_{\lambda})$ with $U'_{\lambda} \subset U_{\lambda}$ but $\partial U'_{\lambda} \cap \partial U_{\lambda} \neq \emptyset$. However, in many cases it is possible to perturb $\partial U'_{\lambda}$, ∂U_{λ} and obtain quadratic-like maps. These “thickening” constructions are well known (see for example [Mi2], [Ly1]) and we will not concentrate on details.

2.5. Small copies of the Mandelbrot set. Small copies of the Mandelbrot set are in one-to-one correspondence with hyperbolic components: for every hyperbolic component $\mathcal{H} \subset \mathcal{M}$, there is a unique small copy $\mathcal{M}_{\mathcal{H}} \supset \mathcal{H}$ so that the canonical homeomorphism of $\mathcal{M}_{\mathcal{H}}$ sends \mathcal{H}

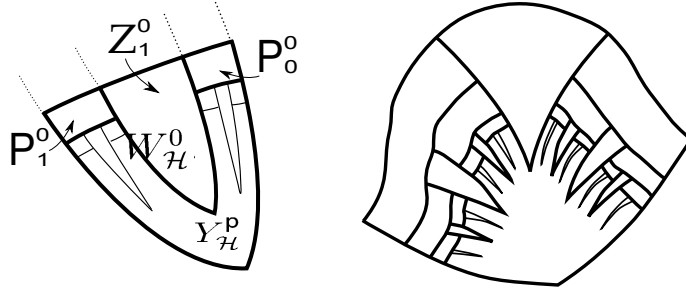


FIGURE 5. Left: the pieces P_0^0 , P_1^0 , Z_1^0 and their pre-images under $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$. Right: a scheme for the decoration tiling: pieces in the decoration tiling are iterated pre-images of P_0^0 , P_1^0 , Z_1^0 under $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$.

to the main hyperbolic component of \mathcal{M} , and every small copy of \mathcal{M} is of this type for a unique component \mathcal{H} .

Consider a hyperbolic component $\mathcal{H} \subset \mathcal{M}$. There are two parameter rays $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ landing at the root of \mathcal{H} , where $a, b \in \mathbb{Q}/\mathbb{Z}$. The angles ϕ_1, ϕ_2 are called the *external angles* of \mathcal{H} . Rays $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ separate the closed sector $\mathcal{W}_{\mathcal{H}}$ from the main hyperbolic component of \mathcal{M} .

The small copy $\mathcal{M}_{\mathcal{H}}$ is in $\mathcal{W}_{\mathcal{H}}$ and is described in the following way.

Let $p \in \mathbb{N}$ be the period of \mathcal{H} ; it means that $2^p \phi_1 = \phi_1$, $2^p \phi_2 = \phi_2$, and p is minimal with this property.

Assume c is in $\text{int}(\mathcal{W}_{\mathcal{H}})$. In the dynamical plane of $f_c(z) = z^2 + c$ the rays R^{ϕ_1}, R^{ϕ_2} land together (Theorem 2.2) and bound the sector $W_{\mathcal{H}}$ (the dynamical counterpart to $\mathcal{W}_{\mathcal{H}}$) containing the critical value c . This sector contains no forward images of R^{ϕ_1} and R^{ϕ_2} . The pre-image $f_c^{-1}(W_{\mathcal{H}})$ is a strip bounded by four rays; two of them are $R^{2^{p-1}\phi_1}$ and $R^{2^{p-1}\phi_2}$. Pulling back this strip along the orbit of the rays R^{ϕ_1} and R^{ϕ_2} we get the strip within $W_{\mathcal{H}}$, we call it *renormalization strip* $Y_{\mathcal{H}}$ (see Figure 4). The parameter renormalization strip $\mathcal{Y}_{\mathcal{H}}$ is defined as the parameter counterpart to $Y_{\mathcal{H}}$.

Let $W_{\mathcal{H}}^0$ be $W_{\mathcal{H}}$ truncated by an equipotential and $Y_{\mathcal{H}}^p$ be the pre-image of $W_{\mathcal{H}}^0$ under $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$; we have $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$ (see Figure 5).

Fact 2.5. *The small copy $\mathcal{M}_{\mathcal{H}}$ is the closure of the set of parameters $c \in \text{int}(\mathcal{Y}_{\mathcal{H}}^p)$ such that in the dynamical plane the critical value c does not escape from $Y_{\mathcal{H}}^p$ under iteration of $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$.*

We will refer to $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$ as the *first renormalization map*. Let us also remark that for a puzzle piece, say \mathcal{Y}_i^n , of a map in \mathcal{V}_1 the upper index usually denotes the *depth* of the puzzle piece; i.e. \mathcal{Y}_i^n is a pre-image under f_c^n of a puzzle piece with depth 0.

2.6. Decoration tiling. A decoration tiling is a certain partition of a neighborhood of a given small copy of the Mandelbrot set. We will



FIGURE 6. Left: the decoration tiling associated with the first renormalization map $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$; the scheme for the tiling is in Figure 5. Right: an example of a generalized decoration tiling; this is a usual tiling for \mathcal{V}_2 .

first describe a decoration tiling consisting of parapuzzle pieces (the left part of Figure 6), after that we will define a generalized decoration tiling (the right part of Figure 6). In all cases we will first define a dynamical partition, its parameter counterpart will be a (generalized) decoration tiling.

Choose a non-parabolic non-Misiurewicz parameter $c \in \mathcal{M}_{\mathcal{H}}$; then c always stays in $\text{int}(Y_{\mathcal{H}}^p)$ under iteration of $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$.

Truncating $\mathcal{W}_{\mathcal{H}}$, $W_{\mathcal{H}}$, and $\mathcal{Y}_{\mathcal{H}}$ at the equipotential at height t , we obtain puzzle pieces $\mathcal{W}_{\mathcal{H}}^0$, $W_{\mathcal{H}}^0$, and $\mathcal{Y}_{\mathcal{H}}^0$; similarly, let $Y_{\mathcal{H}}^p$ be the truncation of $Y_{\mathcal{H}}$ at the equipotential at height $t/2^p$. We decompose $\overline{W_{\mathcal{H}}^0} \setminus Y_{\mathcal{H}}^p$ into topological disks $Z_1^0 \cup P_0^0 \cup P_1^0$ with a puzzle piece $Z_1^0 := \overline{W_{\mathcal{H}}^0} \setminus Y_{\mathcal{H}}^p$ and two “parallelogram” disks P_0^0 and P_1^0 that are the components of $\overline{Y_{\mathcal{H}}^0} \setminus Y_{\mathcal{H}}^p$, see Figure 5. Let T be the set of pullbacks of Z_1^0 , P_0^0 , and P_1^0 under iteration of $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$. The *decoration tiling* $\mathcal{T} = \mathcal{T}(\mathcal{W}_{\mathcal{H}}^0)$ is the set of the parameter counterparts of pieces in T (the left part of Figure 6).

In [Du] we proved (see [PR] for a different proof) the decoration theorem that says, in a strong form, that the diameter of pieces in the decoration tiling tends to 0:

Decoration Theorem. *For any $\varepsilon > 0$, there are at most finitely many pieces in the decoration tiling of $\mathcal{M}_{\mathcal{H}}$ with diameter at least ε .*

Consider a more general situation. Let us include the case $\mathcal{M}_{\mathcal{H}} = \mathcal{M}$ (i.e. \mathcal{H} is the main hyperbolic component of \mathcal{M}) into the consideration. For non-parabolic non-Misiurewicz $c_0 \in \mathcal{M} = \mathcal{M}_{\mathcal{H}}$ we define $W_{\mathcal{H}}^0$ to be the degenerate puzzle piece bounded by only the 0-ray R^0 and truncated by some equipotential. Then $Y_{\mathcal{H}}^1$ is the pre-image of $W_{\mathcal{H}}^0$ under f_{c_0} . For $c \in \text{int}(\mathcal{W}_{\mathcal{H}}^0)$ we have a (degenerate) renormalization map $f_c : Y_{\mathcal{H}}^1 \rightarrow W_{\mathcal{H}}^0$ so that $c \in \mathcal{M}$ if and only if c does not escape from $Y_{\mathcal{H}}^1$ under iteration of $f_c : Y_{\mathcal{H}}^1 \rightarrow W_{\mathcal{H}}^0$.

Let $\mathcal{M}_{\mathcal{H}}$ be a small copy of \mathcal{M} , the case $\mathcal{M} = \mathcal{M}_{\mathcal{H}}$ is allowed, and let c_0 be a non-parabolic non-Misiurewicz point in $\mathcal{M}_{\mathcal{H}}$. Define inductively $Y_{n\mathcal{H}}^{np}$ as the pullback of $Y_{(n-1)\mathcal{H}}^{(n-1)p}$ under $f_{c_0}^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$, where $Y_{1\mathcal{H}}^p = Y_{\mathcal{H}}^p$.

It follows from Fact 2.5 that a parameter $c \in \text{int}(\mathcal{Y}_{n\mathcal{H}}^{np})$ is in $\mathcal{M}_{\mathcal{H}}$ if and only if c does not escape from $Y_{(n+1)\mathcal{H}}^{(n+1)p}$ under iteration of $f_c^p : Y_{(n+1)\mathcal{H}}^{(n+1)p} \rightarrow Y_{n\mathcal{H}}^{np}$. We will refer to $f_c^p : Y_{(n+1)\mathcal{H}}^{(n+1)p} \rightarrow Y_{n\mathcal{H}}^{np}$ as the n -th renormalization map. (Let us note that $\text{int}(\mathcal{Y}_{n\mathcal{H}}^{np}) \setminus \mathcal{M}_{\mathcal{H}}$ consists of finitely many points: one of them is parabolic, all other points are Misiurewicz).

Definition 2.6 (Generalized renormalization map). *Let X^{np} be a generalized puzzle piece that behaves periodically around the Julia set such that ∂X^{np} is isotopic to $\partial Y_{n\mathcal{H}}^{np}$ relative the Julia set. Define $X^{(n+1)p}$ to be a unique pre-image of X^{np} under $f_{c_0}^p$ such that $\partial X^{(n+1)p}$ is isotopic to $\partial Y_{(n+1)\mathcal{H}}^{(n+1)p}$ relative the Julia set.*

If $X^{(n+1)p} \subset X^{np}$ and every component of $\partial X^{(n+1)p} \cap \partial X^{np}$ intersects K_{c_0} , then $f_{c_0}^p : X^{(n+1)p} \rightarrow X^{np}$ is a generalized renormalization map.

It follows that $c_0 \in \text{int}(X^{(n+1)p}) \subset \text{int}(X^{np})$ and so the parameter counterparts \mathcal{X}^{np} and $\mathcal{X}^{(n+1)p}$ of X^{np} and $X^{(n+1)p}$ exist (Property 2.4). For all $c \in \text{int}(\mathcal{X}^{np})$ the map $f_c^p : X^{(n+1)p} \rightarrow X^{np}$ is well defined; and $c \in \mathcal{M}_{\mathcal{H}}$ if and only if c does not escape from $X^{(n+1)p}$ under iteration of $f_c^p : X^{(n+1)p} \rightarrow X^{np}$; in particular, the last map encodes $\mathcal{M}_{\mathcal{H}}$.

It follows that all components of $\overline{X^{np} \setminus X^{(n+1)p}}$ are (up to fixing markings) generalized puzzle pieces. Let T be the set of pre-images of $\overline{X^{np} \setminus X^{(n+1)p}}$ under $f_{c_0}^p : X^{(n+1)p} \rightarrow X^{np}$. The *generalized decoration tiling* \mathcal{T} associated with $f_c^p : X^{(n+1)p} \rightarrow X^{np}$ is the parameter counterpart to T . We have:

Theorem 2.7. *The diameter of pieces in the decoration tiling associated with $f_c^p : X^{(n+1)p} \rightarrow X^{np}$ tends to 0.*

2.7. Compact connected locally connected subsets of the sphere.

Compact connected locally connected sets are characterized by the following theorem, see [Wh, Theorem 4.4, p. 113].

Theorem 2.8. *A connected compact set $K \subset \mathbb{S}^2$ is locally connected if and only if it satisfies the following properties:*

- *For every $\varepsilon > 0$ there are only finitely many connected components of $\mathbb{S}^2 \setminus K$ with diameter at least ε .*
- *The boundary of every connected component of $\mathbb{S}^2 \setminus K$ is locally connected.*

Let us also recall that the continuous image of a compact, connected, locally connected set is locally connected [HY, Theorem 3–22].

Let $K_1, K_2 \subset \mathbb{S}^2$ be compact connected locally connected subsets of the 2-dimensional sphere \mathbb{S}^2 . Let $h : K_1 \rightarrow K_2$ be a homeomorphism

and U be a component of $\mathbb{S}^2 \setminus K_1$. Then h preserves the orientation of U if for every point $a \in U$, every choice of different points $x, y, z \in \partial U$, and non-intersecting curves $\gamma_x, \gamma_y, \gamma_z$ in $U \cup \{x, y, z\}$ connecting a to x, y, z respectively there is a component U' of $\mathbb{S}^2 \setminus K_2$, a point a' in U' , and non-intersecting curves $\gamma'_x, \gamma'_y, \gamma'_z$ in $U' \cup \{h(x), h(y), h(z)\}$ connecting a' to $h(x), h(y), h(z)$ respectively such that the orientation of $\gamma_x, \gamma_y, \gamma_z$ around a is the same as the orientation of $\gamma'_x, \gamma'_y, \gamma'_z$ around a' .

Lemma 2.9. *There is at most one component U' of $\mathbb{S}^2 \setminus K_2$ satisfying the above condition.*

Proof. By planarity, the set $\{h(x), h(y), h(z)\}$ is on the boundaries of at most two components of $\mathbb{S}^2 \setminus K_2$. If $h(x), h(y), h(z) \in \partial U_1$ and $h(x), h(y), h(z) \in \partial U_2$ for $U_1 \neq U_2$, then $h(x), h(y), h(z)$ have different orientations with respect to U_1 and U_2 . \square

A homeomorphism $h : K_1 \rightarrow K_2$ preserves the orientation of the complement if it preserves the orientation of every component of $\mathbb{S}^2 \setminus K_1$. As a corollary of Carathéodory's theorem we get:

Proposition 2.10. *Let $h : K_1 \rightarrow K_2$ be a homeomorphism between compact, connected, locally connected subsets of \mathbb{S}^2 preserving the orientation of the complement. Then h extends to a homeomorphism of \mathbb{S}^2 .*

Proof. By Lemma 2.9 the map h induces the surjective map h' from the set of components of $\mathbb{S}^2 \setminus K_1$ to the set of components of $\mathbb{S}^2 \setminus K_2$. By Carathéodory's theorem for every component U of $\mathbb{S}^2 \setminus K_1$ the map h extends to a homeomorphism from $K_1 \cup U$ onto $K_2 \cup h'(U)$. Applying Carathéodory's theorem for all components of $\mathbb{S}^2 \setminus K_1$ we get a surjective map, call it \tilde{h} , from \mathbb{S}^2 to \mathbb{S}^2 such that $\tilde{h}|_{K_1} = h$ and $\tilde{h}|_{\overline{U}}$ is a homeomorphism for every component U of $\mathbb{S}^2 \setminus K_1$. As the diameter of components in $\mathbb{S}^2 \setminus K_1$ and in $\mathbb{S}^2 \setminus K_2$ tends to 0 the map \tilde{h} is continuous. Therefore, \tilde{h} is a homeomorphism. \square

2.8. Laminations. Laminations were introduced by Thurston to describe connected, locally connected Julia sets of polynomials. Consider the closed unit disc $\overline{\mathbb{D}}$, denote by \mathbb{S}^1 the boundary of $\overline{\mathbb{D}}$; we parametrize \mathbb{S}^1 by angles between 0 and 1 and we denote by “ $<$ ” the cyclic order in \mathbb{S}^1 .

Below we will recall the definition and the construction of a lamination of the Basilica.

Definition 2.11 (A lamination of the Basilica). *A lamination L_B of the Basilica is a collection of curves, called leaves, in $\overline{\mathbb{D}}$ such that*

- every leaf $l \in L$ connects two points of the form $a/(3 \cdot 2^n)$, $b/(3 \cdot 2^n)$ on \mathbb{S}^1 , i.e. $l \cap \mathbb{S}^1 = \{a/(3 \cdot 2^n), b/(3 \cdot 2^n)\}$ and a, b

are coprime to 6; for every point of the form $a/(3 \cdot 2^n)$ there is exactly one leaf that contains this point;

- pairs $\{1/3, 2/3\}$ and $\{5/6, 1/6\}$ are connected by leaves;
- if a pair $\{a, b\} \neq \{1/3, 2/3\}$ is connected by a leaf, then pairs in $\{a, b, a/2, b/2\}$ that are on the same component of $\mathbb{S}^1 \setminus \{1/3, 5/6\}$ are also connected by leaves;
- the diameter of leaves in L_B tends to 0.

Note that this definition gives an algorithm to construct L_B . Let us remark that the last condition in Definition 2.11 is a special property of the Basilica; this implies, in particular, that L_B is a closed set. For a general polynomial the last condition in Definition 2.11 is replaced by weaker conditions.

Two points $a \neq b$ in $\overline{\mathbb{D}}$ are *equivalent* under L_B if they are in the same leaf of L_B . It follows that L_B is a closed equivalence relation and the restriction of L_B to \mathbb{S}^1 is invariant under $z \rightarrow z^2$: if $a, b \in \mathbb{S}^1$ and $a \sim_{L_B} b$, then $a^2 \sim_{L_B} b^2$. Define $K'_B = \overline{\mathbb{D}}/L_B$, $J'_B = \partial K'_B = \mathbb{S}^1/L_B$, and $f'_B : J'_B \rightarrow J'_B$ to be the quotient dynamics:

$$z^2/L_B : \mathbb{S}^1/L_B \rightarrow \mathbb{S}^1/L_B.$$

Extend f'_B to the interior of K'_B such that f'_B has one superattractive periodic cycle with period 2 that attracts all points in $\text{int}(K'_B)$, and locally at the superattractive periodic cycle $f_B'^2$ is conjugate to $z \rightarrow z^2$. It follows from Thurston's theorem that $f'_B : K'_B \rightarrow K'_B$ and $f_B : K_B \rightarrow K_B$ are topologically conjugate.

We will usually denote a leaf $l \in L_B$ by $\langle a, b \rangle$ such that a, b are points in \mathbb{S}^1 connected by l and the length of the arc

$$\{x \in \mathbb{S}^1 : b \geq x \geq a\}$$

is less than $1/2$. For a leaf $l = \langle a/(3 \cdot 2^n), b/(3 \cdot 2^n) \rangle$ with a, b coprime to 6 we call n the *depth* of l .

For simplicity, we will work with z^2 -invariant laminations.

Definition 2.12 (z^2 -invariant laminations). *The lamination L_B of the Basilica is z^2 -invariant if all leaves are simple curves and any leaf $\langle a, b \rangle \neq \langle 1/3, 2/3 \rangle$ is an iterated pre-image of the $\langle 5/6, 1/6 \rangle$ -leaf under $z \rightarrow z^2$.*

An example of z^2 -invariant lamination is in Figure 2.

2.9. Moore's theorem. The following theorem is useful in the mating construction.

Theorem 2.13 (Moore's theorem, [Mu]). *Let \sim be a closed equivalence relation on the 2-sphere \mathbb{S}^2 such that all equivalence classes are connected and non-separating, and not all points are equivalent. Then the quotient space \mathbb{S}^2/\sim is homeomorphic to the 2-sphere.*

3. MATING CONSTRUCTION

Consider a lamination L_B of the Basilica; we assume that L_B is z^2 -invariant. For our convenience, let us complete \mathcal{V}_1 by adding a special point at ∞ so that \mathcal{V}_1 is a Riemann sphere. Then the composition $(1/z) \circ \mathcal{B}$ of the parameter Böttcher function \mathcal{B} and $1/z$ maps $\mathcal{V}_1 \setminus \mathcal{M}$ conformally onto the open unit disc \mathbb{D} .

We denote by

$$\mathcal{L}_B = \bigcup_{l \in L_B} \overline{(1/z \circ \mathcal{B})^{-1}(l|_{\mathbb{D}})}$$

the embedding of L_B into \mathcal{V}_1 . By construction, leaves in \mathcal{L}_B land either at Misiurewicz parameters or at the Basilica parabolic point, see Figure 2.

We keep the same notation for leaves in \mathcal{L}_B as for L_B . This implies that a leaf $l = \langle a, b \rangle \in \mathcal{L}_B$ has the same accesses to the Mandelbrot set as the parameter rays \mathcal{R}^a and \mathcal{R}^b .

The parameter leaf $\langle 1/3, 2/3 \rangle$ surrounds the Basilica limb. Let \mathcal{W}_B be the closed topological disc containing the Basilica limb and bounded by $\langle 1/3, 2/3 \rangle$.

Definition 3.1. *Two point $c_1 \neq c_2$ in \mathcal{V}_1 are equivalent under \mathcal{L}_B if either c_1 and c_2 are in \mathcal{W}_B or c_1 and c_2 are in the same leaf in \mathcal{L}_B .*

3.1. Combinatorics of leaves. Consider a small copy $\mathcal{M}_{\mathcal{H}} \not\subset \mathcal{W}_B$ and let $\mathcal{Y}_{\mathcal{H}}$ be the renormalization strip of $\mathcal{M}_{\mathcal{H}}$. The boundary of $\mathcal{Y}_{\mathcal{H}}$ consists of four rays, call them $\mathcal{R}^{\phi_1}, \mathcal{R}^{\psi_1}, \mathcal{R}^{\psi_2}, \mathcal{R}^{\phi_2}$. We assume that \mathcal{R}^{ϕ_1} and \mathcal{R}^{ϕ_2} are the periodic (*external*) rays of \mathcal{H} and $0 < \phi_1 < \psi_1 < \psi_2 < \phi_2 < 0$ is the cyclic order.

We say \mathcal{L}_B *bounds* $\mathcal{M}_{\mathcal{H}}$ from the right (resp from the left) if there is a leaf $\langle a, b \rangle \in \mathcal{L}_B$ such that $a < \phi_1 < \psi_1 < b$ (resp. $a < \psi_2 < \phi_2 < b$); in particular, the leaf $\langle a, b \rangle$ truncates the strip $\mathcal{Y}_{\mathcal{H}}$ from the right (resp. from the left).

Proposition 3.2. *The copy $\mathcal{M}_{\mathcal{H}}$ as above is bounded from the left (resp. from the right) by \mathcal{L}_B if and only if there is an n such that $5/6 < 2^n \phi_1 < 1/6$ (resp. $5/6 < 2^n \phi_2 < 1/6$).*

Proof. We will prove the case of being bounded from the left; the second case is proved in a similar way.

Moreover, the proposition is equivalent to the following statement involving L_B , where the equivalence follows from the definition of the Böttcher function: there is a leaf $l \in L_B$ separating both ϕ_1, ψ_1 from the center of the unit disc if and only if there is an $n \geq 0$ such that $5/6 < 2^n \phi_1 < 1/6$. Let us prove the last claim.

If there is a leaf $l = \langle a, b \rangle \in L_B$ separating ϕ_1, ψ_1 from the center of the unit disc, then $\langle a, b \rangle \neq \langle 1/3, 2/3 \rangle$ and it follows that there is an n such that $\langle 2^n a, 2^n b \rangle = \langle 5/6, 1/6 \rangle$. By Definition 2.12 we have $5/6 < 2^n \phi_1 < 1/6$.

Suppose now that there is an n such that $5/6 < 2^n \phi_1 < 1/6$. Let us choose the minimal n with the above property; it follows that $n \leq p$, where p is the period of \mathcal{H} ; i.e. $2^p \phi_1 = \phi_1$ and p is minimal.

We claim that $5/6 < 2^n \psi_1 < 1/6$; in other words, $\phi_1, \psi_1 \in \mathbb{S}^1$ escape to the arc bounded by $\langle 5/6, 1/6 \rangle$ in the same pattern. Indeed, the above claim is a consequence of the following trivial statement concerning the combinatorics of dynamic rays: for $c \in \mathcal{M}_{\mathcal{H}} \not\subset \mathcal{W}_B$ the orbit $Y_{\mathcal{H}}, f_c(Y_{\mathcal{H}}), \dots, f_c^{p-1}(Y_{\mathcal{H}})$ of the renormalization strip does not contain the rays $R^{1/6}$ and $R^{5/6}$ (if it did happen, then the sector $\mathcal{W}_{\mathcal{H}}$ would contain $\mathcal{R}^{1/3}$ or $\mathcal{R}^{2/3}$, which would contradict to $\mathcal{M}_{\mathcal{H}} \subset \mathcal{M} \setminus \mathcal{W}_B$).

Pulling back $\langle 5/6, 1/6 \rangle$ -leaf n times by $z \rightarrow z^2$ along the orbit $\phi_1, 2\phi_1, \dots, 2^n \phi_1$ we get a leaf that separates ϕ_1, ψ_1 from the center of the unit disc. \square

Proposition 3.3. *Every copy $\mathcal{M}_{\mathcal{H}} \not\subset \mathcal{W}_B$ is bounded from at least one side. If $\mathcal{M}_{\mathcal{H}}$ is a subcopy of another copy of the Mandelbrot set or is a satellite copy, then $\mathcal{M}_{\mathcal{H}}$ is bounded from both sides.*

Proof. Let $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ be the external rays for \mathcal{H} ; together these rays form the boundary of $\mathcal{W}_{\mathcal{H}}$. Since $\mathcal{H} \not\subset \mathcal{W}_B$, either $0 < \phi_1 < \phi_2 < 1/3$ or $2/3 < \phi_1 < \phi_2 < 1$.

We have:

- if $0 < \phi_1 < \phi_2 < 1/3$, then $5/6 < 0 < \phi_1/2 < \phi_2/2 < 1/6$;
- if $2/3 < \phi_1 < \phi_2 < 1$, then $5/6 < \phi_1/2 + 1/2 < \phi_2/2 + 1/2 < 0 < 1/6$.

Let $p > 0$ be the period of ϕ_1, ϕ_2 ; i.e. $\phi_1 = 2^p \phi_1, \phi_2 = 2^p \phi_2$, and p is the minimal. It is well known (follows from the construction of $Y_{\mathcal{H}}$, see Figure 4 for illustration) that either $\phi_1/2 = 2^{p-1} \phi_1, \phi_2/2 + 1/2 = 2^{p-1} \phi_2$ or $\phi_2/2 = 2^{p-1} \phi_2, \phi_1/2 + 1/2 = 2^{p-1} \phi_1$. It follows from

$$0 < \phi_1/2, \phi_2/2 < 1/6 \quad \text{or} \quad 5/6 < \phi_1/2 + 1/2, \phi_2/2 + 1/2 < 0,$$

and Proposition 3.2 that $\mathcal{M}_{\mathcal{H}}$ is bounded from at least one side.

If $\mathcal{M}_{\mathcal{H}}$ is a satellite copy, then R^{ϕ_1} and R^{ϕ_2} are in one periodic cycle, thus $\mathcal{M}_{\mathcal{H}}$ is bounded from both sides.

Let $\mathcal{M}_{\mathcal{H}'}$ be a subcopy of $\mathcal{M}_{\mathcal{H}}$ and let $\mathcal{R}^{\phi'_1}, \mathcal{R}^{\phi'_2}$ be the external rays of \mathcal{H}' . Consider the dynamical plane of f_c for $c \in \mathcal{M}_{\mathcal{H}'}$.

We have just proved that there is a leaf $\langle a, b \rangle$ that bounds $\mathcal{M}_{\mathcal{H}}$ from at least one side. As $\mathcal{M}_{\mathcal{H}'} \subset \mathcal{M}_{\mathcal{H}}$ the forward orbits of $R^{\phi'_1}, R^{\phi'_2}$ will not escape from the strip $Y_{\mathcal{H}}$ under iteration of $f_c^p : Y_{\mathcal{H}} \rightarrow W_{\mathcal{H}}$. Moreover, the forward orbits of $R^{\phi'_1}$ and $R^{\phi'_2}$ visit both sides of $Y_{\mathcal{H}}$, whence $a < 2^k \phi'_1 < b$ and $a < 2^m \phi'_2 < b$ for some $k, m \geq 0$. This implies that $5/6 < 2^{k+n} \phi'_1 < 1/6$ and $5/6 < 2^{m+n} \phi'_2 < 1/6$ for some $n \geq 0$. Therefore, $\mathcal{M}_{\mathcal{H}'}$ is bounded from both sides (Proposition 3.2). \square

3.2. Decoration tiling associated with \mathcal{L}_B . Let $\mathcal{M}_{\mathcal{H}} \not\subset \mathcal{W}_B$ be a small copy of the Mandelbrot set bounded by $\mathcal{L}_{\mathcal{H}}$ from both sides. The goal of this subsection is to prove the following proposition.

Proposition 3.4. *There is a generalized decoration tiling \mathcal{T} of $\mathcal{M}_{\mathcal{H}}$ such that a component of $\mathcal{V}_1 \setminus \widehat{\mathcal{M}}$ or a leaf in $\mathcal{L}_B \setminus \mathcal{W}_B$ intersects at most three pieces in \mathcal{T} .*

Proof. Let c be a non-parabolic non-Misiurewicz point in $\mathcal{M}_{\mathcal{H}}$. The composition $B_c^{-1} \circ \mathcal{B}$ of Böttcher functions embeds \mathcal{L}_B into the dynamical plane of $f_c = z^2 + c$; denote by

$$L_B^c = B_c^{-1} \circ \mathcal{B}(\mathcal{L}_B) = B_c^{-1} \circ (1/z)(L_B) = \bigcup_{l \in L_B} \overline{(1/z \circ B_c)^{-1}(l|_{\mathbb{D}})}$$

the embedding.

By construction (Definition 2.12), L_B^c has the following description:

$$L_B^c = \langle 1/3, 2/3 \rangle \cup_{n \geq 0} f_c^{-n}(\langle 5/6, 1/6 \rangle).$$

For instance, if l is a leaf of L_B^c and $l \neq \langle 1/3, 2/3 \rangle$, then pre-images of l under f_c^n are also leaves of L_B^c .

Let $\langle a, b \rangle, \langle c, d \rangle$ be the leaves of L_B^c of the maximal depth such that $\langle a, b \rangle, \langle c, d \rangle$ bound the renormalization strip $Y_{\mathcal{H}}$ from the right and left respectively. Consider the second renormalization map $f_c^p : Y_{2\mathcal{H}}^{2p} \rightarrow Y_{\mathcal{H}}^p$ of $\mathcal{M}_{\mathcal{H}}$. By definition, the leaves $\langle a, b \rangle, \langle c, d \rangle$ truncate the strip $Y_{\mathcal{H}} = \lfloor Y_{\mathcal{H}}^p \rfloor$.

We will now construct a generalized renormalization map $f_c^p : X^{2p} \rightarrow X^p$ for $\mathcal{M}_{\mathcal{H}}$ with required properties¹; see Figure 7 for illustration.

Denote by R^{ϕ_1}, R^{ϕ_2} and by R^{ψ_1}, R^{ψ_2} the periodic and pre-periodic rays in $\partial Y_{\mathcal{H}}$; we assume that $0 < \phi_1 < \psi_1 < \psi_2 < \phi_2 < 0$ is the cyclic order.

By assumption, there are points a_0, b_0 on $\langle a, b \rangle$, resp. points c_0, d_0 on $\langle c, d \rangle$, such that $a_0 \in R^{\phi_1}$, $b_0 \in R^{\psi_1}$, and a_0 is between a and b_0 on $\langle a, b \rangle$, resp. $c_0 \in R^{\psi_2}$, $d_0 \in R^{\phi_2}$, and c_0 is between c and d_0 on $\langle c, d \rangle$. We denote by $\langle a_0, b_0 \rangle$, resp. $\langle c_0, d_0 \rangle$, the set of points on $\langle a, b \rangle$, resp. on $\langle c, d \rangle$ that are between a_0 and b_0 , resp. c_0 and d_0 . Define inductively a_n, c_n , resp. b_n, d_n to be the pre-images of a_{n-1} , resp. of d_{n-1} under $f_c^p : Y_{2\mathcal{H}}^{2p} \rightarrow Y_{\mathcal{H}}^p$ such that $a_n \in R^{\phi_1}$ and $c_n \in R^{\psi_2}$, resp. $d_n \in R^{\phi_2}$ and $b_n \in R^{\psi_1}$. Choose simple curves $\alpha_0, \beta_0, \gamma_0, \delta_0$ connecting the pairs $\{a_0, a_1\}, \{b_0, b_1\}, \{c_0, c_1\}, \{d_0, d_1\}$ respectively such that $\alpha_0, \beta_0, \gamma_0, \delta_0$ do not intersect

$$\cup_{n \geq 0} f_c^{-n}(\langle a, b \rangle) \cup_{n \geq 0} f_c^{-n}(\langle c, d \rangle) \cup K_c.$$

Define inductively $\alpha_n \ni a_n$ and $\gamma_n \ni c_n$, resp. $\delta_n \ni d_n$, $\beta_n \ni b_n$ to be the lifts of α_{n-1} , resp. of δ_{n-1} under $f_c^p : Y_{2\mathcal{H}}^{2p} \rightarrow Y_{\mathcal{H}}^p$.

By construction, curves $\cup_i \alpha_i, \cup_i \beta_i, \cup_i \gamma_i, \cup_i \delta_i$ are homotopic to $R^{\phi_1}, R^{\psi_1}, R^{\psi_2}, R^{\phi_2}$ relative the Julia set. Furthermore, $\cup_i \alpha_i, \cup_i \beta_i$ intersect $\langle a, b \rangle$ at a_0, b_0 and $\cup_i \gamma_i, \cup_i \delta_i$ intersect $\langle c, d \rangle$ at c_0, d_0 .

¹If L_B is a lamination as in Figure 2, then it is enough to define X^p to be $Y_{\mathcal{H}}$ truncated by $\langle a, b \rangle$ and $\langle c, d \rangle$

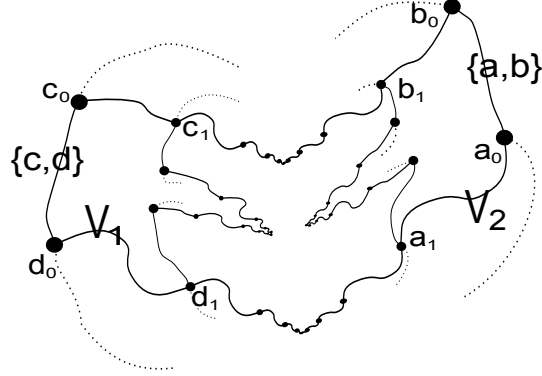


FIGURE 7. Illustration to the Proof of Proposition 3.4. The domain X^p is bounded by the curves $\cup_i \alpha_i$, $\cup_i \beta_i$, $\cup_i \gamma_i$, $\cup_i \delta_i$ and is truncated by the leaves $\langle a, b \rangle$ and $\langle c, d \rangle$. The sets V_1 , V_2 are bounded by $\langle a, b \rangle$, $\langle c, d \rangle$, $f_c^{-p}(\langle a, b \rangle \cup \langle c, d \rangle)$, and K_c ; V_1 , V_2 intersect at most three pieces in the generalized decoration tiling associated with $f_c^p : X^{2p} \rightarrow X^p$.

Define X^p to be the Jordan disk bounded by

$$(\cup_i \alpha_i) \cup \langle a_0, b_0 \rangle \cup (\cup_i \beta_i) \cup (\cup_i \gamma_i) \cup \langle c_0, d_0 \rangle \cup (\cup_i \delta_i);$$

define X^{2p} to be the pre-image of X^p under f_c^p such that $X^{2p} \subset X^p$. We obtain a generalized renormalization map $f_c^p : X^{2p} \rightarrow X^p$.

Let \mathcal{T} be the decoration tiling associated with $f_c^p : X^{2p} \rightarrow X^p$ and T be the dynamical counterpart of \mathcal{T} . Let us show that \mathcal{T} satisfies the claims of the proposition; this is equivalent to the following dynamical version: a component $U \in \mathbb{C} \setminus (K_c \cup L_B^c)$ intersects at most three pieces in T ; a leaf $l \in L_B^c$ intersects at most three pieces in T .

Consider the set

$$S = \mathbb{C} \setminus \left(K_c \bigcup_{n \geq 0} f^{-n}(\langle a, b \rangle \cup \langle c, d \rangle) \right).$$

By definition, U is in a component, say V , of S . It follows that V is bounded by three leaves in $\cup_{n \geq 0} f^{-n}(\langle a, b \rangle \cup \langle c, d \rangle)$ (see Figure 7); and there are exactly two curves in

$$\Gamma = \bigcup_{n \geq 0} f^{-n}(\cup_i \alpha_i) \bigcup_{n \geq 0} f^{-n}(\cup_i \beta_i) \bigcup_{n \geq 0} f^{-n}(\cup_i \gamma_i) \bigcup_{n \geq 0} f^{-n}(\cup_i \delta_i)$$

intersecting V . Recall that pieces in T are bounded by curves in Γ and in $\cup_{n \geq 0} f^{-n}(\langle a, b \rangle \cup \langle c, d \rangle)$; therefore, V intersects at most three pieces in T and so is U .

Consider a leaf l . Then either l is in $\cup_{n \geq 0} f^{-n}(\langle a, b \rangle \cup \langle c, d \rangle)$ or l is in a component, say V , of S . If $l \subset V$, then V intersects at most three pieces in T so the same is true for l . If l is in $\cup_{n \geq 0} f^{-n}(\langle a, b \rangle \cup \langle c, d \rangle)$,

then, by construction, l forms the boundary of two pieces in T and intersects at most one additional piece in T . \square

As corollaries we have:

Corollary 3.5. *The diameter of components in $\mathcal{V}_1 \setminus (\mathcal{W}_B \cup \mathcal{L}_B \cup \mathcal{M})$ tends to 0. The diameter of leaves in $\mathcal{L}_B \setminus \mathcal{W}_B$ tends to 0.*

Proof. By Yoccoz's theorem big pieces (with diameters at least a fixed $\varepsilon > 0$) in $\mathcal{V}_1 \setminus (\mathcal{W}_B \cup \mathcal{L}_B \cup \mathcal{M})$ can not accumulate at finitely renormalizable parameters of the Mandelbrot set. But by Proposition 3.4 and the decoration theorem big components can not accumulate at infinitely renormalizable parameters.

Similarly, the diameter of leaves in $\mathcal{L}_B \setminus \mathcal{W}_B$ tends to 0. \square

Corollary 3.6. *Leaves in $\mathcal{L}_B \setminus \mathcal{W}_B$ can only accumulate at the Mandelbrot set.*

Corollary 3.7. *The equivalence relation \mathcal{L}_B is closed and satisfies the conditions of Moore's theorem.*

The restriction of \mathcal{L}_B to \mathcal{M} is canonical (independent of the choice of L_B).

Proof. By Corollary 3.5 the equivalence relation \mathcal{L}_B is closed. As different leaves of $\mathcal{L}_B \setminus \mathcal{W}_B$ land at different points, equivalence classes in \mathcal{L}_B do not separate points. Finally, points $a, b \in \mathcal{M}$ are equivalent under \mathcal{L}_B if and only if $a = b$, or a, b are the end of a leaf in \mathcal{L}_B (Corollary 3.5), or $a, b \in \mathcal{W}_B$. \square

3.3. The mating $\mathcal{M} \cup_{\mathbb{S}^1} L_B$. Denote by $\mathcal{V}'_2 = \mathcal{V}'_2(\mathcal{L}_B)$ the quotient $\mathcal{V}_1/\mathcal{L}_B$ and by $m : \mathcal{V}_1 \rightarrow \mathcal{V}'_2$ the induced quotient map. By Moore's theorem \mathcal{V}'_2 is topologically a sphere.

Recall that a (continuous) map $m : \mathcal{V}_1 \rightarrow \mathcal{V}'_2$ depends on \mathcal{L}_B , but the restriction $m : \mathcal{M} \rightarrow \mathcal{M}'_2$ is canonical, i.e. it is independent of \mathcal{L}_B .

Define

$$\widehat{\mathcal{M}} := \mathcal{M} \bigcup \mathcal{L}_B \bigcup \mathcal{W}_B$$

(recall that \mathcal{W}_B depends on \mathcal{L}_B); by construction, $m^{-1}(\mathcal{M}'_2) = \widehat{\mathcal{M}}$.

Proposition 3.8. *The following holds:*

- *The boundary of a component of $\mathcal{V}_1 \setminus \widehat{\mathcal{M}}$ contains no infinitely renormalizable parameter.*
- *The boundary of a component of $\mathcal{V}_1 \setminus \widehat{\mathcal{M}}$ is locally connected.*
- *The diameter of components in $\mathcal{V}_1 \setminus \widehat{\mathcal{M}}$ tends to 0. The set $\widehat{\mathcal{M}}$ is locally connected.*
- *The set \mathcal{M}'_2 is locally connected. All spaces $\mathcal{V}'_2 = \mathcal{V}'_2(\mathcal{L}_B)$ are homeomorphic by homeomorphisms preserving \mathcal{M}'_2 .*

Proof. Let \mathcal{E} be a component of $\mathcal{V}_1 \setminus \widehat{\mathcal{M}}$ and $c_0 \in \partial\mathcal{E}$. By Proposition 3.3 every secondary copy of the Mandelbrot set is separated from \mathcal{E} ; thus c_0 is not infinitely renormalizable.

Let us prove local connectivity of $\partial\mathcal{E}$ at c_0 . If $c_0 \notin \mathcal{M}$, then $\partial\mathcal{E}$ is locally connected at c_0 as leaves in \mathcal{L}_B can only accumulate at the Mandelbrot set (Corollary 3.6).

Assume that $c_0 \in \mathcal{M}$. Let us also suppose that c_0 is not the end of a leaf in \mathcal{L}_B and there is exactly one access from \mathcal{E} to c_0 .

Let $\mathcal{P}_1, \mathcal{P}_2, \dots$ be a sequence of parapuzzle pieces shrinking to c_0 [Hu]; i.e. $\mathcal{P}_i \cap \mathcal{M}$ is connected, $c_0 \in \text{int } \mathcal{P}_i$, and $\text{diam}(\mathcal{P}_i) \rightarrow 0$. Let γ_i be the component of $\mathcal{P}_i \cap \partial\mathcal{E}$ containing c_0 . As $\text{diam}(\mathcal{P}_i) \rightarrow 0$ we have $\text{diam}(\gamma_i) \rightarrow 0$.

Let us fix i and show that γ_i is a neighborhood of c_0 in $\partial\mathcal{E}$.

As \mathcal{E} is an open topological disc we have the cyclic order for $\mathcal{L}_B \cap \partial\mathcal{E}$: if $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle, \langle a_3, b_3 \rangle$ are in $\mathcal{L}_B \cap \partial\mathcal{E}$, then up to permuting $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle$:

$$a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < a_1.$$

Let \mathcal{R}^ϕ be the external ray landing at c_0 with the same access to c_0 as \mathcal{E} relative the Mandelbrot set. Consider the set

$$S_\varepsilon = \{\langle a, b \rangle \in \mathcal{L}_B \cap \partial\mathcal{E} : |\phi - a| < \varepsilon \text{ or } |b - \phi| < \varepsilon\}.$$

We will show that leaves in S_ε are in \mathcal{P}_i for sufficiently small ε and if a leaf $\langle a, b \rangle$ is sufficiently close to c_0 in the metric of \mathcal{V}_1 , then $\langle a, b \rangle \in S_\varepsilon$. This will imply that

$$\overline{\bigcup_{l \in S_\varepsilon} l} \subset \gamma_i$$

is a neighborhood of c_0 because leaves in $\mathcal{L}_B \cap \partial\mathcal{E}$ are dense on $\partial\mathcal{E}$.

It follows from triviality of fibers at non infinitely renormalizable parameters that if $\langle a, b \rangle \in S_\varepsilon$ for small ε , then $\langle a, b \rangle$ is close to c_0 in the metric of \mathcal{V}_1 . By Corollary 3.5 the diameter of leaves in \mathcal{L}_B tends to 0. Therefore, there are only finitely many leaves in \mathcal{L}_B that are close to c_0 but are not in \mathcal{P}_i . By choosing sufficiently small ε we get $S_\varepsilon \subset \mathcal{P}_i$.

Assume that there is a sequence $\langle a_i, b_i \rangle$ of leaves in $(\mathcal{L}_B \cap \partial\mathcal{E}) \setminus S_\varepsilon$ accumulating at c_0 . We may assume that a_i tends to $\psi \neq \phi$. Then R^ψ lands at c_0 because the fiber at c_0 is trivial. This contradicts the assumption that there is only one access to c_0 .

It is well known (follows from the branch theorem [Sc1]) that there are at most two accesses to c_0 from $\mathcal{V}_1 \setminus \mathcal{M}$; so there are at most two accesses from \mathcal{E} to c_0 . If there are two accesses², then we can present \mathcal{E} as $\mathcal{E}_1 \cup \mathcal{E}_2$ such that $\mathcal{E}_1, \mathcal{E}_2$ are open topological discs that do not intersect in some neighborhood of c_0 . The same argument as above shows local

²In fact, it would follow from Theorem 1.2 that $\partial\mathcal{E}$ is a Jordan curve as it is so for the boundaries of components in $\mathcal{V}_2 \setminus \mathcal{M}_2$.

connectivity of $\partial\mathcal{E}_1$ and $\partial\mathcal{E}_2$ at c_0 ; this implies local connectivity of $\partial\mathcal{E}$ at c_0 .

If c_0 is an end of a leaf $\langle a, b \rangle \in \mathcal{L}_B \cap \partial\mathcal{E}$, then we can apply the above argument to $\overline{\partial\mathcal{E} \setminus \langle a, b \rangle}$ and conclude local connectivity of $\overline{\partial\mathcal{E} \setminus \langle a, b \rangle}$ at c_0 . This implies local connectivity of $\partial\mathcal{E}$ at c_0 .

We have proved that $\partial\mathcal{E}$ is locally connected. By Corollary 3.5 the diameter of components in $\mathcal{V}_1 \setminus \widehat{\mathcal{M}}$ tends to 0. This implies that $\widehat{\mathcal{M}}$ is locally connected.

Since the continuous image of a compact connected locally connected set is locally connected [HY, Theorem 3–22] we conclude that \mathcal{M}'_2 is locally connected.

Finally, it follows from Proposition 2.10 that \mathcal{V}'_2 is independent of the choice of leaves in L_B up to a homeomorphism preserving \mathcal{M}'_2 . \square

4. PROPERTIES OF \mathcal{V}_2

The second parameter slice \mathcal{V}_2 (Figure 1) is the set of quadratic rational maps that have a periodic critical point of period 2. Every map in \mathcal{V}_2 is conjugate to a unique map of the form

$$(6) \quad g_a(z) = \frac{a}{z^2 + 2z},$$

where $a \in \widehat{\mathbb{C}} \setminus \{0\}$ and the case $a = \infty$ corresponds to the map $1/z^2$.

In normalization (6), infinity is a critical point of period two and -1 is a free critical point (unless $a = \infty$). The set $\mathcal{M}_2 \subset \mathcal{V}_2$ consists of maps such that the free critical point is not in the attracting basin of the $\{\infty, 0\}$ -cycle.

For our convenience, let us complete \mathcal{V}_2 by adding a special point at 0 so that \mathcal{V}_2 is Riemann sphere and \mathcal{M}_2 is a compact subset of \mathcal{V}_2

The map

$$g_1 = \frac{1}{z^2 + 2z}$$

is conjugate to the Basilica polynomial; we denote by $\rho = -1/(z+1)$ the unique Möbius transformation that conjugates g_1 to $f_B = z^2 - 1$:

$$\rho \circ g_1 = f_B \circ \rho.$$

The hyperbolic component containing g_1 is *the main hyperbolic component* of \mathcal{M}_2 ; it is the biggest black component on Figure 1.

4.1. Bubbles. The idea to use bubbles to construct rays is due to Luo [Lu]. He also showed how to define bubble puzzles and parapuzzles.

Let $\Omega = \Omega(a)$ be the attracting basin of the $0-\infty$ -cycle and E_0, E_∞ be the connected (periodic) components of Ω containing $0, \infty$ respectively.

A *bubble* of g_a is a component E of Ω . The generation of a bubble E is the smallest non-negative $n = \text{Gen}(E)$ for which $g_a^n(E) = E_\infty$.

By definition, $-a \in \Omega(a)$ if and only if $a \in \mathcal{V}_2 \setminus \mathcal{M}_2$. It is easy to see that $-a$ can not be in E_0 [AY, Proposition 2.5] because a non-fixed periodic Fatou component can not contain all critical values.

If $-a \in E_\infty$, equivalently $-1 \in E_0$, then $\Omega = E_0 \cup E_\infty$, and the map g_a is quasi-conformally conjugate in a neighborhood of the Julia set to $z \rightarrow z^{-2}$. The last property holds if and only if a is in the main unbounded component on the Figure 1 which will be denoted by \mathcal{E}_∞ .

Furthermore, $-a \in \partial E_\infty$ if and only if $a \in \partial \mathcal{E}_\infty$; this induces a parametrization of $\partial \mathcal{E}_\infty$, see [Ti1].

If $-a \notin E_\infty$ (equivalently $a \notin \mathcal{E}_\infty$), then Ω is a proper superset of $E_0 \cup E_\infty$. Moreover, it follows from the following proposition that $\overline{E_0} \cap \overline{E_\infty} = \alpha = \alpha(g_a)$ is a fixed point of g_a .

Proposition 4.1. *The boundary of each bubble is a locally connected simple closed curve. If $a \notin \mathcal{E}_\infty$, then $\overline{E_0} \cap \overline{E_\infty}$ contains a unique fixed point $\alpha = \alpha(g_a)$.*

Proof. The case $a \in \overline{\mathcal{E}_\infty}$ follows from [Ti1, Theorem B**].

If $a \in \widehat{\mathbb{C}} \setminus \overline{\mathcal{E}_\infty}$, then $\partial E_0 \cup \partial E_\infty$ depends holomorphically on a by the λ -lemma (see [Mc1] for reference). In particular, $\partial E_0, \partial E_\infty$ are locally connected simple closed curves that intersect at a fixed point because it is so for hyperbolic maps in $\widehat{\mathbb{C}} \setminus \overline{\mathcal{E}_\infty}$. Since every bubble E is a preimage of E_∞ we see that ∂E is locally connected simple closed curve. \square

Proposition 4.2. *Assume that $-a \notin E_\infty$. Then there is a unique conformal map $h_a : E_0(a) \cup E_\infty(a) \rightarrow E_0(1) \cup E_\infty(1)$ that conjugates g_a and g_1*

$$(7) \quad \begin{array}{ccc} E_0(a) \cup E_\infty(a) & \xrightarrow{g_a} & E_0(a) \cup E_\infty(a) \\ \downarrow h_a & & \downarrow h_a \\ E_0(1) \cup E_\infty(1) & \xrightarrow{g_1} & E_0(1) \cup E_\infty(1), \end{array}$$

with normalization $h'_a(0) = h'_a(\infty) = 1$. Moreover, h_a extends homeomorphically to the α -fixed point: $h_a(\alpha(g_a)) = \alpha(g_1)$.

Let γ be a curve in

$$\Omega(a) \cup_{n \geq 0} g_a^{-n}(\alpha(g_a))$$

starting at ∞ . If γ does not contain (hit) -1 or its preimage under g_a , then h_a extends uniquely along γ .

Proof. The first part of the proposition follows from the existence of Böttcher coordinates and the definition of $\alpha(g_a)$.

Let us prove the second part. We can present γ as a union $\cup_{n \geq 0} \gamma_{2n}$, where $\gamma_{2n} \subset \gamma_{2n+2}$, such that γ_{2n} is a curve starting at infinity and $g_a^{2n}(\gamma_{2n}) \subset E_0(a) \cup E_\infty(a) \cup \alpha(g_a)$. Then h_a is defined over $g_a^{2n}(\gamma_{2n})$. As γ does not hit -1 or its preimages the map $g_a^{2n} : \gamma_{2n} \rightarrow g_a^{2n}(\gamma_{2n})$ is a covering. Hence h_a extends along γ_{2n} . \square

Assume that $-a \in E_\infty$ but $a \neq \infty$ (i.e. $g_a \neq 1/z^2$). Then there is a unique fixed under g_a^2 internal ray γ_∞ emerging from infinity. If γ_∞ does not hit $-a$, then it lands at a fixed point of J_a ; we define $\alpha(g_a)$ to be the landing point of γ_∞ . If γ_∞ hits $-a$, then we say that $\alpha(g_a)$ is not defined; it will follow from Theorem 4.7 that $\alpha(g_a)$ is defined if and only if the parameter a is not on the parameter ray \mathcal{B}^0 .

Note that if $a \notin \infty$, then h_a is defined in neighborhoods of 0 and ∞ ; if $\alpha(g_a)$ is defined, then h_a extends along $\gamma_\infty \cup g_a(\gamma_\infty) \cup \alpha(g_a)$. It is easy to verify the following statement.

Proposition 4.3. *Assume that $-a \in E_\infty$ and $\alpha(g_a)$ is defined. Then h_a exists in neighborhoods of 0 and ∞ containing $-a$.*

Assume further that γ is a curve in

$$\Omega(a) \cup_{n \geq 0} g_a^{-n}(\alpha(g_a))$$

starting at ∞ such that γ is a concatenation $\gamma(1) \# \gamma(2) \# \dots \# \gamma(n) \# \beta$ or $\gamma(1) \# \gamma(2) \# \gamma(3) \# \dots$, where $\gamma(i)$ is a preimage of γ_∞ and β is a curve in $\Omega(a)$. If γ does not contain (hit) -1 or its preimage under g_a , then h_a extends uniquely along γ .

4.2. Rays in bubbles. Recall that g_1 is conjugate by ρ to the Basilica polynomial. Let γ_0 , resp. γ_∞ , be the straight internal ray in $E_0(1)$, resp. $E_\infty(1)$, that connects 0 and the α -fixed point, resp. ∞ and α . We have:

$$g_1(\gamma_0 \cup \gamma_\infty) = \gamma_\infty \cup \gamma_0.$$

Define T to be

$$\bigcup_{n \geq 0} g_a^{-n}(\gamma_0 \cup \gamma_\infty);$$

it is clear that T is a tree. Furthermore, T intersects $\partial\Omega$ at the pre-images of the α -fixed point; these pre-images will be called \mathfrak{B} -points (if we conjugate g_1 by ρ , then \mathfrak{B} -points are the landing points of the rays $R^{n/(3 \cdot 2^k)}$ for $k \geq 0$ and n coprime to 6). For $x \in T$ we define $\gamma(x)$ to be a unique geodesic within T connecting ∞ and x .

Consider a component E of Ω , denote by $n \geq 0$ the generation of E . Let x be a unique pre-image of ∞ in E and γ'' be an internal ray in E_∞ connecting ∞ and a non- \mathfrak{B} point on ∂E_∞ . Then $g_1^n(x) = \infty$ and γ'' lifts conformally along the orbit of x to the curve, say γ' in E connecting x and a non- \mathfrak{B} point on E ; we call

$$(8) \quad \gamma = \gamma(1) = \gamma' \cup \gamma(x)$$

a *finite ray in bubbles*.

An *infinite ray in bubbles* is an infinite geodesic, say γ , in T :

$$\gamma : [0, \infty) \rightarrow T, \quad \gamma(0) = \infty \quad \text{and} \quad \gamma(t_1) \neq \gamma(t_2) \quad \text{if} \quad t_1 \neq t_2.$$

For g_a define a *finite* (resp. *infinite*) *ray in bubbles* $\gamma(a)$ to be a pre-image of a finite (resp. infinite) ray in bubbles $\gamma(1)$ under h_a provided

that h_a^{-1} extends uniquely along $\gamma(1)$; if this is the case, *we will write* $\gamma = \gamma(1) = \gamma(a)$ and say that γ *exists* for g_a .

The following lemma is a consequence of Propositions 4.2 and 4.3 and the local connectivity of the closure of a bubble (Proposition 4.1).

Lemma 4.4. *A ray in bubbles γ exists for g_a if and only if γ does not hit the critical point -1 or its pre-image. If a finite ray in bubbles exists, then it lands.*

Define

$$G(a) = \{x \in T = T(1) : h_a^{-1} \text{ extends along } \gamma(x)\},$$

and define $T = T(a)$ to be $g_a^{-1}(G(a))$. It follows that T is a tree and any finite ray in bubbles is of the form $\gamma' \cup \gamma(x)$, where x is a pre-image of ∞ , the curve $\gamma(x)$ is a geodesic in $T(a)$ connecting ∞ and x , and γ' is a conformal pre-image of an internal ray in E_∞ .

We say that an angle ϕ is of type “ \mathfrak{B} ” if it is of the form $n/(3 \cdot 2^k)$ such that $k \geq 0$ and n is coprime to 6. We say that a ray R^ϕ or \mathcal{R}^ϕ is of type “ \mathfrak{B} ” (resp. “non- \mathfrak{B} ”) if ϕ is a \mathfrak{B} angle (resp. non- \mathfrak{B} angle). The landing points of \mathfrak{B} (resp. non- \mathfrak{B}) rays will also be called \mathfrak{B} (resp. non- \mathfrak{B}) points.

Recall that $\rho = -1/(z+1)$ conjugates g_1 and f_B .

Definition 4.5 (\mathcal{V}_2 -twins). *Let ϕ be a non- \mathfrak{B} angle. A ray in bubbles γ is the \mathcal{V}_2 -twin of an external ray R^ϕ if $\rho(\gamma)$ and $R^{-\phi}$ land at the same point in J_B .*

For every ray with non- \mathfrak{B} angle there is a unique \mathcal{V}_2 -twin. From now on we will write $\gamma = B^\phi$ if γ is the \mathcal{V}_2 -twin of R^ϕ . The angle of $\gamma = B^\phi$ is ϕ . The following lemma follows directly from the definition.

Lemma 4.6. *Every ray in bubbles is a \mathcal{V}_2 -twin of a ray of maps in \mathcal{V}_1 .*

4.3. Parameter rays in bubbles. Assume $-1 \in E \subset \Omega$, where $E \neq E_\infty$, and $n = \text{Gen}(E)$. Then h_a extends uniquely to

$$\bigcup_{\text{Gen}(E) < n} \overline{E}.$$

In particular, $h_a(-a)$ is defined and canonical (unique). If $-a \in E_\infty$, then $h_a(-a)$ is defined and canonical by Proposition 4.3.

It is well known that for components $\mathcal{E}' \neq \mathcal{E}''$ of $\mathcal{V}_2 \setminus \mathcal{M}_2$ the closures $\overline{\mathcal{E}'}$, $\overline{\mathcal{E}''}$ either do not intersect or $\overline{\mathcal{E}'} \cap \overline{\mathcal{E}''}$ is a Misiurewicz parameter; which will be called a \mathfrak{B} Misiurewicz parameter; \mathfrak{B} Misiurewicz parameters in \mathcal{V}_2 are the matings of \mathfrak{B} Misiurewicz polynomials (i.e. the corresponding parameters are the landing points of \mathfrak{B} parameter rays) in $\mathcal{M} \setminus \mathcal{W}_B$ and the Basilica polynomial.

Theorem 4.7. *The map*

$$(9) \quad a \rightarrow \rho(h_a(-a))$$

is a homeomorphism between

$$\Sigma = \bigcup_{\mathcal{E} \subset V_2 \setminus \mathcal{M}_2} \overline{\mathcal{E}}$$

(the union is taken over all components in $V_2 \setminus \mathcal{M}_2$) and

$$\bigcup_{E \in \text{int}(K_B) \setminus \{1/2\text{-limb}\}} \overline{E}$$

(the union is taken over all components in $\text{int}(K_B) \setminus \{1/2\text{-limb}\}$).

Furthermore, let $c(\phi) \in \mathcal{M} \setminus \mathcal{W}_B$ be the landing point of a \mathfrak{B} ray \mathcal{R}^ϕ . Then the mating $f_{c(\phi)} \cup_{\mathbb{S}^1} f_B$ is in Σ and the image of $f_{c(\phi)} \cup_{\mathbb{S}^1} f_B$ under (9) is the landing point of \mathcal{R}^ϕ .

Proof. This theorem is well known. The (difficult) case $a \in \overline{\mathcal{E}_\infty}$ follows from [Ti1, Theorem C]. The case $a \in \Sigma \setminus \overline{\mathcal{E}_\infty}$ follows from the implicit function theorem (it is straightforward that (9) is invertible for $a \in \Sigma \setminus \overline{\mathcal{E}_\infty}$).

The second part of the theorem follows from the definitions of \mathcal{V}_2 -twins and matings. \square

In particular, it follows from Theorem 4.7 that the boundary of a component in $\mathcal{V}_2 \setminus \mathcal{M}_2$ is a locally connected simple closed curve.

For a non- \mathfrak{B} angle ϕ the pre-image of B^ϕ under (9) is the *parameter ray in bubbles* \mathcal{B}^ϕ . We say that \mathcal{B}^ϕ is the *parameter counterpart* to B^ϕ and is the \mathcal{V}_2 -*twin* of \mathcal{R}^ϕ .

4.4. Rational rays. Rationally accessible points. Ray portraits. We say that a ray is *rational* if it is either periodic or pre-periodic. A point in the dynamical plane is *rationally accessible* if it is either a parabolic or repelling periodic point or a pre-image of a parabolic or repelling periodic point. A parameter in \mathcal{V}_1 or in \mathcal{V}_2 is *rationally accessible* if it is either a parabolic or Misiurewicz parameter.

Let x be a rationally accessible point of a map in \mathcal{V}_1 . Then the *ray portrait* P_x of x is the set of rays landing at x . Similarly, we define the *ray portrait* P_x^B of a rationally accessible point x of a map in \mathcal{V}_2 .

We say that a ray portrait is a *non- \mathfrak{B}* portrait if it does not contain a \mathfrak{B} ray.

The *parameter ray portrait* \mathcal{P}_c of a rationally accessible parameter $c \in \mathcal{M}$ is the set of parameter rays landing at c . Similarly, we define the *parameter ray portrait* \mathcal{P}_a^B of a rationally accessible parameter a in \mathcal{M}_2 . A portrait \mathcal{P}_c is *non- \mathfrak{B}* if it does not contain a \mathfrak{B} ray. Note that for $c \in \mathcal{V}_1 \setminus \mathcal{W}_B$ any \mathfrak{B} portrait contains exactly one ray.

We will show later that the set of non- \mathfrak{B} parameter portraits in $\mathcal{V}_1 \setminus \mathcal{W}_B$ is in bijection preserving angles with the set of parameter portraits in \mathcal{V}_2 (Theorem 5.1).

4.5. Bubble puzzle and parapuzzle. A *bubble puzzle piece* $Y^B(a)$ is a closed topological disc in the dynamical plane of g_a bounded by rational landing rays in bubbles such that the forward orbit of $\partial Y^B(a)$ under g_a does not intersect $\text{int}(Y^B(a))$. By analogy to \mathcal{V}_1 we define $[Y^B(a)]$ to be the union of all rays that form the boundary of $Y^B(a)$; because there is no equipotential in \mathcal{V}_2 we have $\partial Y^B(a) \subset [Y^B(a)]$.

Two bubble puzzle pieces $Y^B(a')$, $Y^B(a'')$ are *combinatorially equivalent* (simply, $Y^B = Y^B(a') = Y^B(a'')$) if the composition $h_{a''}^{-1} \circ h_{a'}$ extends to $[Y^B(a')]$ and induces a homeomorphism between $\partial Y^B(a')$ and $\partial Y^B(a'')$ except at finitely many points in $\partial Y^B(a')$, $\partial Y^B(a'')$ that are the landing points of rays in bubbles. Equivalently, there is a homeomorphism between $\partial Y^B(a')$ and $\partial Y^B(a'')$ preserving the angles of rays.

A closed disc \mathcal{Y}^B bounded by rational landing parameter rays in bubbles is a *parameter bubble puzzle piece* if \mathcal{Y}^B is a parameter counterpart to a puzzle piece, denoted by Y^B , in the following sense: Map (9) induces a homeomorphism between $\partial \mathcal{Y}^B$ and ∂Y^B except finitely many points as above.

We say that a bubble parapuzzle piece \mathcal{Y}^B is the \mathcal{V}_2 -*twin* of a parapuzzle piece \mathcal{Y} in \mathcal{V}_1 if $\partial \mathcal{Y}^B$ is combinatorially equivalent to $\partial \mathcal{Y}$ in the following sense: a pair $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$ of landing together rays is in $[\mathcal{Y}^B]$ if and only if a pair $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ of landing together rays is in $[\mathcal{Y}]$. Note that the height of the equipotential truncating \mathcal{Y} is irrelevant for the definition of \mathcal{Y}^B .

Similarly, the \mathcal{V}_2 -twins of dynamic puzzle pieces are defined. *We will use an upper index “B” to indicate the \mathcal{V}_2 -twins of puzzle or parapuzzle pieces.*

5. LANDING OF RAYS IN \mathcal{V}_2

The goal of this section is to prove Theorem 5.1 which is a \mathcal{V}_2 -analogy of Theorem 2.2.

Assume that rational rays $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$ land together. We denote by $\mathcal{Y}^B(\phi_1, \phi_2)$ the parameter disc bounded by $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$ and not containing 0; later we will show that $\mathcal{Y}^B(\phi_1, \phi_2)$ is a parapuzzle piece. Similarly, $Y^B(\phi_1, \phi_2)$ is the puzzle piece bounded by B^{ϕ_1}, B^{ϕ_2} and not containing 0 provided that B^{ϕ_1}, B^{ϕ_2} land together and the forward orbit of B^{ϕ_1}, B^{ϕ_2} does not intersect $\text{int}(Y^B(\phi_1, \phi_2))$. By definition, $\mathcal{Y}^B(\phi_1, \phi_2)$ is the parameter counterpart to $Y^B(\phi_1, \phi_2)$ if $Y^B(\phi_1, \phi_2)$ exists.

We say that two ray portraits are *equivalent* if the angles of rays in these portraits are in bijection. We say that two sets S_1, S_2 of ray portraits *are in bijection preserving angles* if there is a bijection $\omega : S_1 \rightarrow S_2$ such that for every $t \in S_1$ the portraits $t, \omega(t)$ are equivalent.

Theorem 5.1 (Correspondence theorem in \mathcal{V}_2). *Every rational parameter ray in bubbles lands. Furthermore:*

- A rational ray in bubbles B^ϕ lands in the dynamical plane of g_a if and only if $a \notin \cup_{n>0} \mathcal{B}^{2^n\phi}$.
- If rational rays $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$ land together, then $a \in \text{int}(\mathcal{Y}^B(\phi_1, \phi_2))$ if and only if the dynamic rays B^{ϕ_1}, B^{ϕ_2} land together and $-a \in \text{int}(Y^B(\phi_1, \phi_2))$.
- Parameter non- \mathfrak{B} ray portraits in $\mathcal{V}_1 \setminus \mathcal{W}_B$ and in \mathcal{V}_2 are in bijection preserving angles.
- If a rational non- \mathfrak{B} ray \mathcal{R}^ϕ lands at $c \in \mathcal{M} \setminus \mathcal{W}_B$, then B^ϕ lands at $m(c)$, where the map $c \rightarrow m(c)$ is defined in Definition 5.11.

Corollary 5.2. A bubble parapuzzle piece \mathcal{Y}_t^B exists if and only if \mathcal{Y}_t exists in $\mathcal{V}_1 \setminus \mathcal{W}_B$. A dynamic puzzle piece X_n^B exists in $\text{int}(\mathcal{Y}_t^B)$ if and only if X_n exists in $\text{int}(\mathcal{Y}_t)$.

Proof. Follows from Theorem 5.1 and the observation that a bubble parapuzzle piece \mathcal{Y}_t^B is of the form

$$\overline{\mathcal{Y}^B(\phi_1, \phi_2) \setminus (\cup_i \mathcal{Y}^B(\psi_{1,i}, \phi_{2,i}))}.$$

□

The proof of Theorem 5.1 is organized as follows. We will first prove a general result that every rationally accessible point is the landing point of at least one rational ray in a tree of pre-images (Theorem 5.3). Then we will deduce that every rationally accessible point of a map in $\mathcal{M}_2 \setminus \{\mathfrak{B} \text{ Misiurewicz parameters}\}$ is the landing point of at least one rational ray in bubbles (Proposition 5.4).

Further, thanks to mating constructions, Theorem 5.1 holds for hyperbolic components and Misiurewicz parameters (Subsection 5.3).

We say that a rationally accessible point x of a map in \mathcal{V}_2 is *transitive* if all rays in bubbles landing at x are in the same grand orbit. Similarly, we define the transitivity for rationally accessible points of maps in \mathcal{V}_1 .

A rationally accessible parameter x in \mathcal{M} or in \mathcal{M}_2 is *transitive* if the angles of the rays landing at x are in the same grand orbit under the doubling map.

A rational ray (parameter or dynamic) is *transitive* (resp. *intransitive*) if it lands at a transitive (resp. non-transitive or *intransitive*) point.

In Subsection 5.4 we will prove Theorem 5.1 for the transitive case. This will allow us to consider partitions of \mathcal{V}_2 by transitive rational rays. Using this we will show that Theorem 5.1 holds for the intransitive case.

5.1. Trees of pre-images. Trees of pre-images are used in [Ne] to give a symbolic description of the Julia sets of sub-hyperbolic maps.

Consider a rational map $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with degree $d \geq 2$ and let P_g be the postcritical set of g . Choose a base point “*” in $\widehat{\mathbb{C}} \setminus P_g$; for

simplicity, let us assume that $*$ is not periodic. Let

$$V = \bigcup_{n \geq 0} g^{-n}(*)$$

be the set of pre-images of $*$. If $v \in g^{-n}(*)$, then we say that the *depth* of v is n . For each $*_i$ in $g^{-1}(*)$ choose a curve γ_i in $\widehat{\mathbb{C}} \setminus P_g$ connecting $*$ and $*_i$. Define

$$E = \bigcup_{n \geq 0} g^{-n}(\cup_{i=0}^{d-1} \gamma_i)$$

to be the set of pre-images of $\cup_{i=0}^{d-1} \gamma_i$. The *tree of pre-images* $T = T(*, \{\gamma_i\})$ consists of the vertex set V and the edge set E . It follows that T is a d -rooted tree.

A *ray* r in T is an infinite geodesic $\cup_{n \geq 0} \gamma^{(n)}$ starting at $*$, where $\gamma^{(n)}$ is an edge in T . The *image* of r is the ray $g(r) = \cup_{n \geq 0} \gamma^{(n+1)}$. A ray r is *periodic* if $g^p(r) = r$ for some $p > 0$.

Theorem 5.3. *Every periodic (resp. preperiodic) rationally accessible point is the landing point of at least one periodic (resp. preperiodic) ray in T .*

Proof. Let x be a periodic point with period $p \geq 1$. Let us suppose that x is a repelling point. Then there is a neighborhood U of x so that the dynamics g^p in U is conjugate to $z \rightarrow \lambda z$, where $|\lambda| > 1$. By Montel's theorem there is an $m > 0$ such that $g^{pm}(U)$ contains T . It follows that U contains at least one vertex $v \in V$ with depth pm .

Let α_0 be the geodesic in T starting at $g^{pm}f(v) = *$ and ending at v . Define inductively α_n to be the lift of α_{n-1} under

$$g^{pm} : U \rightarrow g^{pm}(U) \supset T$$

to the end of α_{n-1} . It follows that $\alpha_i \subset U$ for $i > 0$ and the diameter of α_i tends exponentially to 0 because of local expansion at x . We conclude that $\cup_{i \geq 0} \alpha_i$ lands at x ; it is obvious that $\cup_{i \geq 0} \alpha_i$ is a periodic ray (α_i is a preimage of α_{i-1}).

If x is a parabolic point, then choose U to be a repelling petal attached to x . The same argument as above shows that x is the landing point of a periodic ray.

Assume that x is a preperiodic rationally accessible point. Then for some $n > 0$ the point $g^n(x)$ is a periodic rationally accessible point. We have just proved that there is a periodic ray r in T landing at $g^n(x)$. The lift of r to x is a preperiodic ray landing at x . \square

Recall that a rationally accessible point is non- \mathfrak{B} if it is not a preperiodic point that is on the boundary of a bubble.

Proposition 5.4. *Let g_a be a map in $\mathcal{M}_2 \setminus \{\mathfrak{B} \text{ Misiurewicz parameters}\}$. Then every periodic (resp. preperiodic) rationally accessible non- \mathfrak{B} point is the landing point of at least one periodic (resp. preperiodic) ray in bubbles.*

Proof. Consider a rationally accessible point $x \in J_a$. The case when x is on the boundary of a bubble follows from local connectivity of the boundary of a bubbles (Proposition 4.1). Let us assume that x is not on the boundary of a bubble.

Choose a base point $*$ close to ∞ . Then two pre-images $g_a^{-1}(*)$ are close to 0 and -2 . Denote by E_∞, E_0, E_{-2} the bubbles containing $\infty, 0, -2$ respectively. Choose connections

$$\gamma_0 \subset \overline{E_\infty} \cup \overline{E_0} \text{ and } \gamma_1 \subset \overline{E_\infty} \cup \overline{E_{-2}}$$

from $*$ to $g_a^{-1}(*)$ so that γ_0, γ_1 coincide with rays in bubbles except in small neighborhoods of $\{\infty\}$, $\{0\}$, and $\{-2\}$. By Theorem 5.3 there is a ray R in $T = T(*, \gamma_0, \gamma_1)$ that lands at x . We will now show that there is a ray in bubbles that is “close” to R and landing at x .

Consider h_a as in Proposition 4.2; by the assumption on g_a the map h_a extends to

$$h_a : \bigcup_{E \subset \Omega(a)} \overline{E} \rightarrow \bigcup_{E \subset \Omega(1)} \overline{E},$$

where the unions are taken over all components in $\Omega(a)$ and $\Omega(1)$ respectively, and h_a is a homeomorphism over $\cup_{E \subset \Omega(1)} E \cup \{\mathfrak{B}\text{-Misiurewicz parameters}\}$. Then $h_a(T)$ is a tree of preimages in the dynamical plane of g_1 . As g_1 is hyperbolic (it is conjugate to the Basilica polynomial) all rays in $h_a(T)$ lands (see more details in [Ne]). Denote by x' the landing point of $h_a(R)$; it is clear that x' is rationally accessible and is not on the boundary of a bubble; otherwise x would also be on the boundary of a bubble.

By definition, $h_a(R)$ is of the form $\cup_i \alpha_i$ such that α_i connects vertices, say $*_{i-1}$ and $*_i$, of $h_a(T)$ with depth $i-1$ and i . Denote by $E(i)$ the bubble containing $*_i$. As $h_a(R)$ is rational there is $m > 0$ and $p > 0$ such that for all $n > m$ the map g_1^p maps $E(n+p)$ onto $E(m)$. It follows that every bubble is at most finitely many times in the sequence $E(0), E(1), E(2), \dots$; otherwise x' would be on the boundary of a bubble that is infinitely many times in the above sequence.

Let B^ϕ be a ray in bubbles landing at x' ; this ray exists by the hyperbolicity of g_1 . It is clear that B^ϕ is in $\cup_i \overline{E(i)}$. Moreover, we can choose a disc $D(i) \subset E(i)$ and an annulus $A(i) \subset E(i)$ such that B^ϕ is in $h_a(R) \cup_i D(i)$, the annulus $A(i)$ surrounds $D(i)$, and the modulus of $A(i)$ is at least a fixed $\varepsilon > 0$. Indeed, we choose some $D(1), \dots, D(m)$ and $A(1), \dots, A(m)$ and we define $D(n+p)$, resp. $A(n+p)$ to be a conformal pullback of $D(n)$, resp. $A(n)$.

Let us show that $h_a^{-1}(B^\phi)$ lands at x . By construction,

$$h_a^{-1}(B^\phi) \subset R \cup_i h_a^{-1}(D(i))$$

so it is enough to show that $\text{diam}(h_a^{-1}(D(i)))$ tends to 0.

Observe that the distance between $h_a^{-1}(D(i))$ and J_a tends to 0; otherwise $\widehat{\mathbb{C}}$ would have an infinite area. But $h_a^{-1}(D(i))$ is separated

from J_a by $h_a^{-1}(A(i))$ with modulus at least $\varepsilon > 0$. This implies that the diameter of $h_a^{-1}(D(i))$ tends to 0. \square

5.2. Properties of dynamic rays in bubbles. Recall that periodic and preperiodic points of g_a are solutions of algebraic equations, so these points depends holomorphically on a unless they collide with other periodic or pre-periodic points (the parabolic case).

Lemma 5.5 (Repelling ray portraits are stable). *Let B^ϕ be a rational ray in bubbles landing at $x(a')$ such that $x(a')$ is a periodic or preperiodic repelling point of $g_{a'}$, then there is a neighborhood \mathcal{V} of the parameter a' such that for all $a \in \mathcal{V} \setminus \cup_{i \geq 1} \mathcal{B}^{2^i \phi}$ the ray B^ϕ lands at $x(a)$.*

Proof. Assume that $x(a)$ is a repelling periodic point with period p and let $U(a)$ be a neighborhood of x so that the dynamics g_a^p is conjugate to $z \rightarrow \lambda_a z$ in $U(a)$. For $a = a'$ there is a pre-image $b(a')$ of ∞ such that $b(a') \in B^\phi \cap U(a')$. The last property is stable under small perturbation unless $a \in \cup_{i \geq 1} \mathcal{B}^{2^i \phi}$. Therefore, if $a \notin \cup_{i \geq 1} \mathcal{B}^{2^i \phi}$ is close to a' , then B^ϕ lands at $x(a)$ as there is a local expansion in $U(a) \ni x(a)$.

The preperiodic case follows from the periodic case. \square

Lemma 5.6. *For a rational angle ϕ the closure of B^ϕ depends holomorphically on a in $\mathcal{V}_2 \setminus \cup_{i \geq 1} \overline{\mathcal{B}^{2^i \phi}}$.*

Proof. By construction, B^ϕ moves holomorphically in $\mathcal{V}_2 \setminus \cup_{i \geq 1} \overline{\mathcal{B}^{2^i \phi}}$. By the λ -lemma (see [Mc1] for reference) the motion of B^ϕ extends to its closure. \square

Lemma 5.7. *Let a be a parameter in \mathcal{M}_2 . If $\overline{\cup_{n \geq 0} g_a^n(-a)}$ does not intersect any ray in bubbles, then all rational rays in bubbles land in the dynamical plane of g_a .*

Proof. By assumption the map

$$g_a : \widehat{\mathbb{C}} \setminus g_a^{-1} \left(\overline{\cup_{n \geq 0} g_a^n(-a)} \cup 0 \cup \{\infty\} \right) \rightarrow \widehat{\mathbb{C}} \setminus \left(\overline{\cup_{n \geq 0} g_a^n(-a)} \cup 0 \cup \{\infty\} \right)$$

is a covering of hyperbolic surfaces while

$$\widehat{\mathbb{C}} \setminus g_a^{-1} \left(\overline{\cup_{n \geq 0} g_a^n(-a)} \cup 0 \cup \{\infty\} \right) \hookrightarrow \widehat{\mathbb{C}} \setminus \left(\overline{\cup_{n \geq 0} g_a^n(-a)} \cup 0 \cup \{\infty\} \right)$$

is a contracting inclusion.

Let B^ψ be a periodic ray in bubbles. Then it is of the form $\cup_{i \geq 0} \alpha_i$ such that α_{i+1} is a pre-image of α_i under $g_{a_0}^p$. As α_2 is disjoint from $\overline{\cup_{n \geq 0} g_a^n(-a)} \cup 0 \cup \{\infty\}$ the curve α_2 has a finite hyperbolic diameter in $S = \widehat{\mathbb{C}} \setminus \left(\overline{\cup_{n \geq 0} g_a^n(-a)} \cup 0 \cup \{\infty\} \right)$. Hence the hyperbolic diameter of α_i in S is bounded and, moreover, if B^ψ does not accumulate at $\overline{\cup_{n \geq 0} g_a^n(-a)}$, then the hyperbolic diameter of α_i tends to 0. This implies that the Euclidean diameter of α_i tends to 0. We conclude that every accumulation point of B^ψ is a solution of $g_{a_0}^p(z) = z$. As the set of

solutions is totally disconnected the accumulation set of B^ψ is one point; i.e. B^ψ lands.

The preperiodic case follows from the periodic case. \square

5.3. Hyperbolic components and Misiurewicz parameters. Let $\mathcal{H} \subset \mathcal{V}_1 \setminus \mathcal{W}_B$ be a hyperbolic component. The \mathcal{V}_2 -*twin* \mathcal{H}^B of \mathcal{H} is the set of matings of polynomials in \mathcal{H} with the Basilica polynomial. It is known that all hyperbolic components of \mathcal{M}_2 are \mathcal{V}_2 -twins of hyperbolic components in \mathcal{V}_1 .

For a hyperbolic component $\mathcal{H} \in \mathcal{M}$ parameters in $\overline{\mathcal{H}}$ are parametrized by the multiplier $\lambda_{\mathcal{H}}$ of the associated with \mathcal{H} non-repelling periodic cycle. Similarly, \mathcal{H}^B is parametrized by the *multiplier* $\lambda_{\mathcal{H}^B}$. The *canonical homeomorphism*

$$(10) \quad c \rightarrow m_{\mathcal{H}}(c), \quad c \in \overline{\mathcal{H}}, \quad m_{\mathcal{H}}(c) \in \overline{\mathcal{H}^B}$$

is defined so that it preserves the multiplier: $\lambda_{\mathcal{H}}(c) = \lambda_{\mathcal{H}^B}(m(c))$.

Lemma 5.8 (The hyperbolic case). *Non- \mathfrak{B} rational ray portraits of $c \in \mathcal{H}$ and rational ray portraits of $m_{\mathcal{H}}(c) \in \mathcal{H}^B$ are in bijection preserving angles.*

Proof. Follows from the mating construction as it preserves non- \mathfrak{B} ray portraits. \square

Corollary 5.9. *Different hyperbolic components in \mathcal{M}_2 have different sets of the ray portraits of rationally accessible points.*

Proof. Follows from Lemma 5.8 and the well known fact that different components in $\mathcal{M} \setminus \mathcal{W}_B$ have different sets of non- \mathfrak{B} ray portraits. \square

The *root* h of a hyperbolic component \mathcal{H}^B is a unique point h on $\partial\mathcal{H}^B$ such that $\lambda_{\mathcal{H}^B}(h) = 1$. The following proposition says that the parabolic bifurcation in \mathcal{M}_2 is the same as in $\mathcal{M} \setminus \mathcal{W}_B$.

Proposition 5.10. *Let $m(h)$ be a parabolic parameter in \mathcal{V}_2 .*

- (A) *There is a unique hyperbolic component \mathcal{H}^B of \mathcal{M}_2 such that $m(h)$ is the root of \mathcal{H}^B .*

Let \overline{q} be the parabolic cycle in the dynamical plane of $m(h)$. Assume that p is the period of \overline{q} and rp is the number of attracting parabolic petals attached to \overline{q} .

- (B) *In the dynamical planes of maps in \mathcal{H}^B there is a bounded attracting cycle \overline{y} (i.e. $\overline{y} \neq \{\infty, 0\}$) with period rp and there is a repelling periodic cycle \overline{x} with period p such that:*
- *points in \overline{x} are on the boundaries of periodic Fatou components that form the immediate attracting basin of \overline{y} ;*
 - *\overline{x} (of maps in \mathcal{H}^B) and \overline{q} (of the map $g_{m(h)}$) have the same ray portraits.*

- (C) If $a \in \mathcal{H}^B$ tends to $m(h)$, then \bar{x} and \bar{y} tends to \bar{q} .
- (D) Rational ray portraits of $m(h)$ and of maps in \mathcal{H}^B are in bijection preserving angles.
- (E) If $r = 1$, then \mathcal{H}^B is a unique hyperbolic component of \mathcal{M}_2 containing $m(h)$ on its boundary.

Denote by h be the root of \mathcal{H} .

- (F) Repelling non- \mathfrak{B} ray portraits of h and $m(h)$ are in bijection preserving angles.
- (G) Parabolic non- \mathfrak{B} ray portraits of h and $m(h)$ are in bijection preserving angles.

Assume that $r > 1$ and let $\lambda_{\bar{q}}$ be the multiplier of \bar{q} . Then:

- (H) There is a unique hyperbolic component \mathcal{H}_2^B of \mathcal{M}_2 such that $m(h) \in \overline{\mathcal{H}_2^B}$ and $\lambda_{\bar{q}} = \lambda_{\mathcal{H}_2^B}(m(h))$.
- (I) In the dynamical planes of maps in \mathcal{H}_2^B there is a bounded attracting cycle \bar{x} (i.e. $\bar{x} \neq \{\infty, 0\}$) with period p and there is a repelling periodic cycle \bar{y} with period rp such that
 - points in \bar{y} are on the boundaries of periodic Fatou components that form the immediate attracting basin of \bar{x} ;
 - the set of rays landing at points in \bar{y} (of maps in \mathcal{H}_2^B) is in bijection preserving angles with the set of rays landing at points in \bar{q} (of the map $g_{m(h)}$).
- (J) The parameter h is on the boundary of \mathcal{H}_2^B .
- (K) The components \mathcal{H}^B and \mathcal{H}_2^B are the only hyperbolic components in \mathcal{M}_2 containing $m(h)$ on their boundaries.

Proof. The proof repeats arguments of the proof of [Mi1, Theorem 4.1], see also [Mi1, Lemma 4.4]), so we give only the sketch of the proof; and we will skip the calculations involving Taylor series.

Let us first verify (A), (B), (C), (D), (H), (I). Parabolic points of a rational map g are multiple roots of equations $g^n(z) = z$; or, in other words, parabolic cycles are collisions of periodic cycles. If g is a map in \mathcal{V}_2 , then it has at most one parabolic cycle that splits into two periodic cycles under a small perturbation because there is only one free critical point.

Let \bar{q} , p , r be as in the conditions of the proposition. Let \mathcal{U} be a sufficiently small neighborhood of $m(h)$. For $a \in \mathcal{U} \setminus \{m(h)\}$ the cycle \bar{q} splits into two cycles with periods p and rp , call them \bar{x} and \bar{y} ; we assume that \bar{x} has period p and \bar{y} has period rp . Note that if $r = 1$, then \bar{x} and \bar{y} interchange when a circles $m(h)$; i.e. \bar{x} , \bar{y} are not distinguishable.

Let $\lambda_{\bar{x}}(a)$, resp. $\lambda_{\bar{y}}(a)$ be the multiplier of \bar{x} , resp. \bar{y} . When a tends to $m(h)$ the multiplier $\lambda_{\bar{x}}(a)$ tends to $\lambda_{\bar{q}}$ while $\lambda_{\bar{y}}(a)$ tends to 1. As $\lambda_{\bar{y}}(a)$ is a holomorphic function it attains all values sufficiently close to 1. Therefore, we can find a parameter path $\gamma^+(t)$ in \mathcal{U} , where $t \in [0, \delta]$, starting at $m(h)$ such that $\lambda_{\bar{y}}(\gamma(t))$ is real and is greater

than 1 for $t > 0$. In the dynamical planes of $g_{\gamma^+(t)}$ attracting parabolic petals attached to \bar{q} become attracting Fatou components, the cycle \bar{y} is attracting, and ray portraits of \bar{q} become ray portraits of \bar{x} .

Define \mathcal{H}^B to be the component containing parameters on $\gamma^+(t)$ for $t > 0$. Parts (B), (C) follows from the construction. By Lemma 5.7 all rays in bubbles land in the dynamical plane of $g_{m(h)}$ because $\overline{\cup_{n \geq 0} g_{m(h)}^n(-m(h))} \subset \{\text{attracting petals}\} \cup \bar{q}$. This implies that all ray portraits are stable on $\gamma^+(t)$: parabolic ray portraits are stable by Part (B) while repelling portraits are stable by Lemma 5.5. This shows Part (D). If h is also the root of a hyperbolic component \mathcal{H}'^B , then maps in \mathcal{H}^B and in \mathcal{H}'^B have the same ray portraits; it follows from Corollary 5.9 that $\mathcal{H}'^B = \mathcal{H}^B$. This finishes the proof of Part (A).

Assume that $r > 1$; equivalently, $\lambda_{\bar{q}} \neq 1$. Then we can find a parameter path $\gamma^-(t)$ in \mathcal{U} , where $t \in [0, \delta]$, starting at $m(h)$ such that $\lambda_{\bar{q}}(\gamma(t))$ is real and is less than 1 for $t > 0$. In the dynamical planes of $g_{\gamma^-(t)}$ groups of attracting parabolic petals attached to the same point in \bar{q} become attracting Fatou components, the cycle \bar{x} is attracting, and rays in bubbles landing at points in \bar{q} land at points in \bar{y} . Define \mathcal{H}_2^B to be the component containing parameters on $\gamma^-(t)$ for $t > 0$. Parts (H), (I) are verified as above.

For maps in \mathcal{H}^B the periodic cycle \bar{x} is the unique periodic cycle with period p on the boundaries of periodic Fatou components that form the immediate attracting basin of \bar{y} . Let $\bar{x}^{(1)}$ be the unique periodic cycle with period p on the boundaries of periodic bounded Fatou components of maps in \mathcal{H} . By the mating construction ray portraits of \bar{x} and $\bar{x}^{(1)}$ are in bijection preserving angles. Furthermore, periodic parabolic ray portraits of h are in bijection preserving angles with ray portraits of $\bar{x}^{(1)}$ for maps in \mathcal{H}_2 . Therefore, Parts (F), (G) follows from Part (B).

If $r > 1$, then h is also on the boundary of a hyperbolic component $\mathcal{H}'_2 \neq \mathcal{H}$. It follows that \mathcal{H}'_2 and \mathcal{H}_2 have the same ray portraits. Therefore, $\mathcal{H}'_2 = \mathcal{H}_2$ (Part J).

If a hyperbolic component \mathcal{H}'^B contains $m(h)$ on the boundary, then ray portraits of maps in \mathcal{H}'^B either in bijection preserving angles with ray portraits of maps in \mathcal{H}^B or in bijection preserving angles with ray portraits of maps in \mathcal{H}_2^B . Therefore, by Corollary 5.9 either $\mathcal{H}'^B = \mathcal{H}^B$ or $\mathcal{H}'^B = \mathcal{H}_2^B$ (Parts (E) and (K)). \square

It follows from Proposition 5.10 that the following definition is correct (does not depend on the choice of $m_{\mathcal{H}}$).

Definition 5.11 (The map $c \rightarrow m(c)$). *If c is in the closure of a hyperbolic component \mathcal{H} of $\mathcal{M} \setminus \mathcal{W}_B$, then $m(c) = m_{\mathcal{H}}(c) \in \overline{\mathcal{H}^B}$ is defined as in (10).*

If $c \in \mathcal{M} \setminus \mathcal{W}_B$ is a Misiurewicz parameter, then $m(c)$ is the mating of f_c and f_B .

Lemma 5.12 (Non- \mathfrak{B} Misiurewicz case). *Let $c \in \mathcal{V}_1 \setminus \mathcal{W}_B$ be a non- \mathfrak{B} Misiurewicz parameter and $g_{m(c)} = f_c \cup_{\mathbb{S}^1} f_B$ be the mating. Then non- \mathfrak{B} ray portraits of f_c and $g_{m(c)}$ are in bijection preserving angles.*

Proof. Follows from the mating construction as it preserves non- \mathfrak{B} ray portraits. \square

Lemma 5.13 (\mathfrak{B} Misiurewicz case). *Let $c \in \mathcal{V}_1 \setminus \mathcal{W}_B$ be a \mathfrak{B} Misiurewicz parameter and $g_{m(c)} = f_c \cup_{\mathbb{S}^1} f_B$ be the mating. Consider a ray portrait P_y . The ray portrait P_y^B exists for $g_{m(c)}$ if and only if*

$$m(c) \notin \bigcup_{R^\phi \in P_y} \bigcup_{n>0} \mathcal{B}^{2^n \phi}.$$

Proof. The portrait P_y survives during the mating if and only if for every R^ϕ in P_y the ray in bubbles B^ϕ exists in the dynamical plane of $g_{m(c)}$. Recall that B^ϕ exists if and only if it does not hit a pre-image of $-a$. Equivalently, B^ϕ exists if and only if there is no $n > 0$ such that $-a \in B^{2^n \phi}$. This is equivalent to the condition stated in the lemma. \square

Consider

$$\Sigma = \bigcup_{\mathcal{E} \subset V_2 \setminus \mathcal{M}_2} \bar{\mathcal{E}},$$

as in Theorem 4.7 (the union is taken over all components in $V_2 \setminus \mathcal{M}_2$). By definition, Σ is an infinite tree of closed discs; note that Σ is neither closed nor open. Recall that all parameter rays in bubbles are in Σ , start at infinity, and have the cyclic order around infinity. For rational angles ϕ_1, ϕ_2 with $0 < \phi_1 < \phi_2 < 1$ we denote by $\Sigma(\phi_1, \phi_2)$ the set of parameters in Σ that are between the rays \mathcal{B}^{ϕ_1} and \mathcal{B}^{ϕ_2} in the anti-clockwise direction. If $\mathcal{Y}^B(\phi_1, \phi_2)$ exists, then $\Sigma(\phi_1, \phi_2) = \Sigma \cap \text{int}(\mathcal{Y}^B(\phi_1, \phi_2))$.

Lemma 5.14 (Theorem 5.3 holds for Σ). *Assume rays $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ land together. Then the following are equivalent:*

- $a \in \text{int}(\Sigma(\phi_1, \phi_2))$.
- $\mathcal{Y}^B(\phi_1, \phi_2)$ exists for g_a and $-a \in \text{int}(\mathcal{Y}^B(\phi_1, \phi_2))$.

Proof. Consider a component \mathcal{U} of $\text{int}(\Sigma) \setminus (\bigcup_{n \geq 0} \mathcal{B}^{2^n \phi_1} \cup \bigcup_{n \geq 0} \mathcal{B}^{2^n \phi_2})$. We need to verify the claim for maps in \mathcal{U} . Note that maps in \mathcal{U} are hyperbolic so all rays in bubbles that exist land. We only need to verify that the rays land in expected patterns.

Choose a \mathfrak{B} Misiurewicz parameter $m(c) \in \partial \mathcal{U} \setminus (\bigcup_{n \geq 0} \mathcal{B}^{2^n \phi_1} \cup \bigcup_{n \geq 0} \mathcal{B}^{2^n \phi_2})$. By the stability of periodic rays (Lemma 5.5) it is sufficient to verify the claim for $m(c)$.

By the second part of Theorem 4.7 we have:

$$(A) \ m(c) \in \Sigma(\phi_1, \phi_2)$$

if and only if

(B) $c \in \text{int}(\mathcal{Y}(\phi_1, \phi_2))$.

It follows from Lemma 5.13 that the following are equivalent:

- (C) $Y^B(\phi_1, \phi_2)$ exists for $g_{m(c)}$ and $-m(c) \in \text{int}(Y^B(\phi_1, \phi_2))$;
- (D) $Y(\phi_1, \phi_2)$ exists for f_c and $c \in \text{int}(Y(\phi_1, \phi_2))$.

By the correspondence theorem for \mathcal{V}_1 (Theorem 2.2) Claims (B) and (D) are equivalent. Therefore, (A) is equivalent to (C). \square

5.4. The transitive case.

Proposition 5.15 (Theorem 5.3, the transitive case). *Assume that transitive rays $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ with rational angles land together in $\mathcal{V}_1 \setminus \mathcal{W}_B$. Then $\mathcal{Y}^B(\phi_1, \phi_2)$ exists in \mathcal{V}_2 and for all $a \in \text{int} \mathcal{Y}^B(\phi_1, \phi_2)$ the bubble puzzle piece $Y^B(\phi_1, \phi_2)$ exists and $-a \in \text{int}(Y^B(\phi_1, \phi_2))$.*

If h is the landing point of $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$, then $m(h)$ is the landing point of $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$.

Proof. We will adopt arguments in [Mil].

Assume first that ϕ_1, ϕ_2 are periodic angles. Denote by \mathcal{X} the set of parameters $a \in \mathcal{V}_2$ such that the rays B^{ϕ_1}, B^{ϕ_2} land together in the dynamical plane of g_a , the common landing point $\gamma(a)$ is transitive repelling, and $-a \in \text{int}(Y^B(\phi_1, \phi_2))$. It is easy to see that \mathcal{X} is open. We need to show that the rays $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$ land together at $m(h) \in \mathcal{V}_2$ and $\mathcal{X} = \text{int}(\mathcal{Y}^B(\phi_1, \phi_2))$.

Observe that $\text{int}(\mathcal{X})$ and $\text{int}(\mathcal{V}_2 \setminus \mathcal{X})$ are not empty. Indeed, it follows from Lemma 5.8 that a hyperbolic component \mathcal{H} is in $\mathcal{Y}(\phi_1, \phi_2)$ if and only if \mathcal{H}^B is in \mathcal{X} . Hence the claim follows from the easy fact that there are hyperbolic components inside as well as outside of $\mathcal{Y}(\phi_1, \phi_2)$.

Consider a point $a_0 \in \partial \mathcal{X}$. By definition, one of the following holds:

- (A) $\gamma(a_0)$ is not a repelling point;
- (B) $Y^B(\phi_1, \phi_2)$ does not exist;
- (C) $Y^B(\phi_1, \phi_2)$ exists but $-a_0 \notin \text{int}(Y^B(\phi_1, \phi_2))$;
- (D) $\gamma(a_0)$ is not transitive.

If **Case (A)** occur, then a_0 is on the boundary of a hyperbolic component. There are finitely many hyperbolic components such that the associated non-repelling cycle has the same period as $\gamma(a)$. Therefore, we can write $a_0 \in \cup_{i=1}^n \partial \overline{\mathcal{H}_i^B}$. Denote by \mathcal{I} the set of indifferent parameters (Siegel or Cremer parameters) on $\cup_{i=1}^n \partial \overline{\mathcal{H}_i^B}$. Then either:

- (A1) $a_0 \in \mathcal{I}$; or
- (A2) a_0 is a parabolic parameter.

Note that \mathcal{I} is totally disconnected because it is totally disconnected subset of a closed set $\cup_{i=1}^n \partial \overline{\mathcal{H}_i^B}$; we will exclude Case (A1) later.

Consider **Case (A2)**. Let \bar{q} be the parabolic periodic cycle in the dynamical plane of g_{a_0} ; by the assumption, $\gamma(a_0)$ is in \bar{q} (more precisely, \bar{q} is a collision of the periodic cycle containing $\gamma(a_0)$ with another periodic cycle). By Lemma 5.7 all rational rays in bubbles land in the

dynamical plane of g_{a_0} because $\overline{\cup_{n \geq 0} g_{a_0}^n(-a_0)} \subset \{\text{attracting petals}\} \cup \overline{q}$. If B^{ϕ_1} does not land at $\gamma(a_0) \in \overline{q}$, then it lands at a periodic repelling point and, by the stability of repelling portraits (Lemma 5.5), B^{ϕ_1} does not land at $\gamma(a)$ in a neighborhood of a_0 . This contradicts the assumption that $a_0 \in \partial\mathcal{X}$. Therefore, B^{ϕ_1} (and, similarly, B^{ϕ_2}) lands at $\gamma(a_0) \in \overline{q}$. We claim that there is a unique parameter with the above properties.

Let us write $a_0 = m(c_0)$ as in Definition 5.11 so that c_0 is the root of a hyperbolic component, say \mathcal{H} . By Proposition 5.10, Part (G) the rays R^{ϕ_1} and R^{ϕ_2} land together at a parabolic periodic point in the dynamical plane of f_{c_0} . Then R^{ϕ_1} , R^{ϕ_2} separate 0 and the critical value c_0 because this property depends only on the combinatorics of R^{ϕ_1} , R^{ϕ_2} . This implies that c_0 is the landing point of the parameter rays \mathcal{R}^{ϕ_1} and \mathcal{R}^{ϕ_2} by the correspondence theorem in \mathcal{V}_1 . In the inverse direction: by Lemma 5.8 the component \mathcal{H}^B is in \mathcal{X} and so its root $a_0 = m(c_0)$ is on $\partial\mathcal{X}$ by Proposition 5.10, Part (B). As R^{ϕ_1} , R^{ϕ_2} are transitive \mathcal{H} is not primitive. Therefore, there is a component $\mathcal{H}_2 \neq \mathcal{H}$ such that $c_0 \in \overline{\mathcal{H}_2}$. It follows that $\mathcal{H}_2 \not\subset \mathcal{Y}(\phi_1, \phi_2)$; this implies that $\mathcal{H}_2^B \subset \mathcal{V}_2 \setminus \overline{\mathcal{X}}$. We get $a_0 = m(c_0) \in \partial(\mathcal{V}_2 \setminus \overline{\mathcal{X}})$.

We denote by $m(h) = a_0 = m(c_0)$ the unique point in Case (A2).

Consider **Case (B)**. If $Y^B(\phi_1, \phi_2)$ does not exist, then either:

- (B1) at least one of the rays B^{ϕ_1} , B^{ϕ_2} does not exist;
- (B2) B^{ϕ_1} , B^{ϕ_2} exist but exactly one of the rays B^{ϕ_1} , B^{ϕ_2} does not land at $\gamma(a_0)$;
- (B3) B^{ϕ_1} , B^{ϕ_2} exist but both rays B^{ϕ_1} , B^{ϕ_2} do not land at $\gamma(a_0)$;

By Lemma 4.4 one of the rays B^{ϕ_1} , B^{ϕ_2} does not exist if and only if $a_0 \in \mathcal{B}^{2^n \phi_1} \cup \mathcal{B}^{2^n \phi_2}$. This describes **Case (B1)**.

Let us show that **Cases (B2), (B3), (C), and (D)** do not occur. The rays B^{ϕ_1} , B^{ϕ_2} are in one periodic orbit and so if one of them lands at $\gamma(a_0)$, then so does another one. This excludes Case (B2).

Assume B^{ϕ_1} , B^{ϕ_2} do not land at $\gamma(a_0)$. Then by Proposition 5.4 there is a ray B^ϕ landing at $\gamma(a_0)$. By Lemma 5.5 the ray B^ϕ lands at $\gamma(a)$ in some neighborhood of a_0 . This contradicts the assumption that $a_0 \in \partial\mathcal{X}$ because in \mathcal{X} the ray B^ϕ does not land at $\gamma(a)$. This excludes Case (B3). Similarly, Case (D) is excluded: if $\gamma(a_0)$ is not transitive, then there is a new ray B^ϕ landing at $\gamma(a_0)$ and, by Lemma 5.5, the ray B^ϕ lands at $\gamma(a)$ in a neighborhood of a_0 which is the contradiction of $a_0 \in \partial\mathcal{X}$.

If $Y^B(\phi_1, \phi_2)$ exists but $-a_0 \notin \text{int}(Y^B(\phi_1, \phi_2))$, then $-a_0 \in \partial Y^B(\phi_1, \phi_2)$. Observe that $-a \neq \gamma(a)$ because $\gamma(a)$ is periodic. Thus $-a_0 \in B^{\phi_1} \cup B^{\phi_2}$, but then these rays do not exist because they hit singularities. This excludes Case (C).

We have shown that

$$\partial\mathcal{X} \subset \cup_{n \geq 0} \mathcal{B}^{2^n \phi_1} \cup \cup_{n \geq 0} \mathcal{B}^{2^n \phi_2} \cup \{m(h)\} \cup \mathcal{I},$$

where \mathcal{I} and $m(h)$ are defined in Case (A). By Lemma 5.14 the set

$$(\cup_{n \geq 0} \mathcal{B}^{2^n \phi_1} \cup_{n \geq 0} \mathcal{B}^{2^n \phi_2}) \setminus (\mathcal{B}^{\phi_1} \cup \mathcal{B}^{\phi_2})$$

can intersect $\partial \mathcal{X}$ only at \mathfrak{B} Misiurewicz parameters. Therefore:

$$\partial \mathcal{X} \subset \mathcal{B}^{\phi_1} \cup \mathcal{B}^{\phi_2} \cup \{m(h)\} \cup \{\mathfrak{B} \text{ Misiurewicz parameters}\} \cup \mathcal{I}.$$

Also, by Lemma 5.14, we have $\mathcal{B}^{\phi_1} \cup \mathcal{B}^{\phi_2} \subset \partial \mathcal{X}$. This implies that the accumulation sets of \mathcal{B}^{ϕ_1} and \mathcal{B}^{ϕ_2} are in $\partial \mathcal{X}$. Observe that the parameter rays can not accumulate on points in hyperbolic components. We conclude that the accumulation sets of \mathcal{B}^{ϕ_1} and \mathcal{B}^{ϕ_2} are in $\mathcal{L} = \{m(h)\} \cup \{\mathfrak{B} \text{ Misiurewicz parameters}\} \cup \mathcal{I}$. Note that \mathcal{L} is totally disconnected as it is a union of the totally disconnected set $\mathcal{I} \subset \cup_{i=1}^n \partial \overline{\mathcal{H}}_i^B$ and a countable set that is disjoint from $\cup_{i=1}^n \partial \overline{\mathcal{H}}_i^B$. We conclude that $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$ land at points in \mathcal{L} . Recall that $m(h) \in \partial \mathcal{X}$ and $m(h) \in \partial(\mathcal{V}_2 \setminus \overline{\mathcal{X}})$. This can happen only if $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$ land at $m(h)$.

We obtain $\overline{\mathcal{X}} = \mathcal{Y}^B(\phi_1, \phi_2)$. Moreover, for $a \in \text{int}(\mathcal{Y}^B(\phi_1, \phi_2))$ the rays B^{ϕ_1}, B^{ϕ_2} land together and $-a \in \text{int}(Y^B(\phi_1, \phi_2))$.

The proof in the **pre-periodic case** is similar to the periodic case, so we will only explain the differences. Assume that ϕ_1, ϕ_2 are pre-periodic with pre-period s . Let us use the induction on s .

As above denote by \mathcal{X} the set of parameters $a \in \mathcal{V}_2$ such that the rays B^{ϕ_1}, B^{ϕ_2} land together in the dynamical plane of g_a , the common landing point $\gamma(a)$ is transitive preperiodic repelling, and $-a \in \text{int}(Y^B(\phi_1, \phi_2))$.

It is well known (follows from Theorem 2.2) that there are periodic angles ψ_1, ψ_2 in the forward orbit of ϕ_1, ϕ_2 so that $\mathcal{R}^{\psi_1}, \mathcal{R}^{\psi_2}$ land together and separate $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ from 0. We may assume that $0 < \psi_1 < \phi_1 < \phi_2 < \psi_2 < 0$ is the cyclic order. Further, if $\psi_1 < 2^k \phi_1 < \psi_2$ for $s > k > 0$, then $\psi_1 < 2^k \phi_2 < \psi_2$, the rays $\mathcal{R}^{2^k \phi_1}, \mathcal{R}^{2^k \phi_2}$ land together, and, by the induction assumption, the rays $\mathcal{B}^{2^k \phi_1}, \mathcal{B}^{2^k \phi_2}$ land together in \mathcal{V}_2 . Denote by \mathcal{Y}^B the closure of the connected component of

$$\mathcal{V}_2 \setminus \left(\bigcup_{\psi_1 < 2^k \phi_1 < \psi_2} \left(\mathcal{B}^{2^k \phi_1} \cup \mathcal{B}^{2^k \phi_2} \right) \cup (\mathcal{B}^{\psi_2} \cup \mathcal{B}^{\psi_1}) \right)$$

containing the accumulation set of \mathcal{B}^{ϕ_1} . It follows that $\mathcal{X} \subset \mathcal{Y}^B$ and for all $a \in \text{int}(\mathcal{Y}^B)$ the puzzle $Y^B(\phi_1, \phi_2)$ exists.

Consider $a_0 \in \partial \mathcal{X}$; as in the periodic case we have Cases (A,1–2), (B,1–3), (C), (D). Case (A) does not occur. Case (B1) occurs only if $a_0 \in (\mathcal{B}^{\phi_1} \cup \mathcal{B}^{\phi_2}) \cap (\mathcal{B}^{\psi_1} \cup \mathcal{B}^{\psi_2})$. Cases (B2), (B3), and (D) do not occur because $Y^B(\phi_1, \phi_2)$ exists in \mathcal{Y}^B . Case (C) occurs if and only if $a_0 \in (\mathcal{B}^{\phi_1} \cup \mathcal{B}^{\phi_2}) \setminus (\mathcal{B}^{\psi_1} \cup \mathcal{B}^{\psi_2})$ or a_0 is the Misiurewicz parameter that is the mating of the Basilica polynomial and the common landing point of $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$. In the last case a_0 is also on the boundary of $\mathcal{V}_2 \setminus \overline{\mathcal{X}}$. \square

5.5. More on small copies of the Mandelbrot set. Recall that a small copy $\mathcal{M}_{\mathcal{H}} \subset \mathcal{M}$ is primitive if \mathcal{H} is not attached to a hyperbolic component of smaller period. Equivalently, the root of $\mathcal{M}_{\mathcal{H}}$ is not transitive. We need the following well known fact about \mathcal{V}_1 .

Proposition 5.16. *Let $\mathcal{M}_{\mathcal{H}} \subset \mathcal{M}$ be a primitive copy of the Mandelbrot set and let p be the period of the renormalization for $\mathcal{M}_{\mathcal{H}}$. Then there is a sequence \mathcal{Y}_n of parapuzzle pieces bounded by transitive rays such that:*

- $\mathcal{Y}_n \subset \text{int}(\mathcal{Y}_{n-1})$;
- $\mathcal{M}_{\mathcal{H}} = \cap_n \mathcal{Y}_n$;
- if $c \in \mathcal{Y}_{n-1}$, then $f_c^p : Y_{n-1} \rightarrow Y_n$ is a quadratic like map.

Proof. The statement is well known so we will give the sketch of the proof.

Consider c_0 in $\mathcal{M}_{\mathcal{H}}$. Let K_1, \dots, K_p be the cycle of small filled in Julia sets associated with $\mathcal{M}_{\mathcal{H}}$; we assume that $c_0 \in K_1$.

Let $\mathcal{M}_{\mathcal{H}'} \neq \mathcal{M}_{\mathcal{H}}$ be the smallest copy of \mathcal{M} containing $\mathcal{M}_{\mathcal{H}}$. If $\mathcal{M}_{\mathcal{H}}$ is maximal, then take $\mathcal{M}_{\mathcal{H}'} := \mathcal{M}$. Let \bar{p} be the periodic cycle associated with \mathcal{H}' ; i.e. \bar{p} is attracting if and only if the parameter is in \mathcal{H}' (if $\mathcal{M}_{\mathcal{H}'} = \mathcal{M}$, then \bar{p} is the α -fixed point).

In the dynamical plane of f_{c_0} the cycle \bar{p} is repelling. Then there is a puzzle piece Y_1 bounded by rays landing at points in \bar{p} or their preimages such that $K_1 \subset Y_1$ but $K_i \cap Y_1 = \emptyset$ for $i \neq 1$.

Define $Y_n \supset K_1$ to be the pullback of Y_1 under $f_{c_0}^{(n-1)p}$ along the orbit of K_1 . The sequence \mathcal{Y}_n satisfies the conditions of the proposition. \square

Proposition 5.17. *Let \mathcal{Y}_n be a sequence as in Proposition 5.16 such that $\mathcal{Y}_n \not\subset \mathcal{W}_B$. Then for all $a \in \cap_n \mathcal{Y}_n^B$ all rational rays in bubbles land in the dynamical plane of g_a .*

Proof. By Proposition 5.15 the \mathcal{V}_2 -twins \mathcal{Y}_n^B exist and $g_a^p : Y_{n-1}^B \rightarrow Y_n^B$ is quadratic like for $a \in \mathcal{Y}_{n-1}^B$. Hence for $a_0 \in \cap_n \mathcal{Y}_n^B$ the map g_{a_0} has a cycle K_1^B, \dots, K_p^B of connected small filled in Julia sets.

Observe that rays in bubbles do not intersect $\cup_i K_i^B$. Indeed, $\cup_i K_i^B$ does not intersect the attracting basin of the ∞ -0 cycle so it is sufficient to verify that $\cup_i K_i^B$ does not contain $\alpha(g_a)$ (the fixed point $\overline{E_\infty} \cap \overline{E_0}$). This is so because Y_n does not contain the landing points of $R^{1/3}, R^{2/3}$.

The proposition follows from Lemma 5.7 because $\cup_i K_i^B \supset \overline{\cup_{n \geq 0} g_a^n(-a)}$. \square

Lemma 5.18. *If a rational non- \mathfrak{B} ray \mathcal{R}^ϕ lands in a parapuzzle piece \mathcal{Y} and \mathcal{Y}^B exists, then the accumulation set of \mathcal{B}^ϕ is in \mathcal{Y}^B .*

If a rational non- \mathfrak{B} ray R^ϕ lands in a puzzle piece Y in the dynamical plane of f_c and if Y^B and B^ϕ exist in the dynamical plane of g_a , then the accumulation set of B^ϕ is in Y^B .

Proof. If \mathcal{R}^ϕ lands in \mathcal{Y} , then either \mathcal{R}^ϕ is in $\partial[\mathcal{Y}]$ or \mathcal{R}^ϕ lands in $\text{int}(\mathcal{Y})$. In the first case \mathcal{B}^ϕ is in $\partial[\mathcal{Y}^B]$ and the claim is obvious. In the second case there is a moment when \mathcal{B}^ϕ enters $\text{int}(\mathcal{Y}^B)$; after that \mathcal{B}^ϕ always stays in $\text{int}(\mathcal{Y}^B)$.

Similarly, the second (dynamical) part of the proposition is verified. \square

5.6. The intransitive case.

Proposition 5.19. *Proposition 5.15 holds for intransitive rays R^{ϕ_1} , R^{ϕ_2} .*

Proof. The proof is similar to the proof of Proposition 5.15 so we will explain only the differences.

Assume ϕ_1, ϕ_2 are periodic. Denote by \mathcal{X} the set of parameters $a \in \mathcal{V}_2$ such that the rays B^{ϕ_1}, B^{ϕ_2} land together in the dynamical plane of g_a , the common landing point $\gamma(a)$ is repelling, and $-a \in \text{int}(Y^B(\phi_1, \phi_2))$. It is easy to see that \mathcal{X} is open. Consider a point $a_0 \in \partial\mathcal{X}$. By definition, one of the following holds:

- (A) $\gamma(a_0)$ is not a repelling point;
- (B) $Y^B(\phi_1, \phi_2)$ does not exist;
- (C) $Y^B(\phi_1, \phi_2)$ exists but $-a_0 \notin \text{int}(Y^B(\phi_1, \phi_2))$.

Furthermore, Case (A) is subdivided into Cases (A1), (A2) and Case (B) is subdivided into Cases (B1)–(B3); see the proof of Proposition 5.15.

Cases (A1), (A2), (B1), (B3), and (C) are the same as in Proposition 5.15. Let us now show that Case (B2) does not occur. Recall that we consider the parameter $a_0 \in \partial\mathcal{X}$; without loss of generality we may assume that B^{ϕ_1} does not land at $\gamma(a_0)$. It follows that B^{ϕ_1} does not land at any point because landing is stable under small perturbations (Lemma 5.5).

Note that a_0 is in the accumulation set of a ray $\mathcal{B}^{2^n\phi_1}$ for some $n \geq 0$; otherwise $\overline{B^{\phi_1}}$ would be stable by Lemma 5.6. Choose a parapuzzle piece \mathcal{Y}_t bounded by transitive rays such that $\mathcal{Y}(\phi_1, \phi_2) \subset \mathcal{Y}_t$ and \mathcal{Y}_t does not contain rays in $\cup_{n \geq 0} \mathcal{R}^{2^n\phi_1} \setminus \{\mathcal{R}^{\phi_1}\}$.

By Proposition 5.15 the \mathcal{V}_2 -twin \mathcal{Y}_t^B exists and, moreover, $\mathcal{X} \subset \mathcal{Y}_t^B$. By Lemma 5.18 the parapuzzle piece \mathcal{Y}_t^B does not contain accumulation set of $\mathcal{B}^{2^n\phi_1} \neq \mathcal{B}^{\phi_1}$. Therefore, a_0 is in $\overline{\mathcal{B}^{\phi_1}}$.

Denote by $h \in \mathcal{V}_1$ the landing point of \mathcal{R}^{ϕ_1} . As R^{ϕ_1} is intransitive h is the root of a small primitive copy \mathcal{M}_h of the Mandelbrot set. Let \mathcal{Y}_n be a sequence as in Proposition 5.16 for \mathcal{M}_h . Then $a_0 \in \cap_n \mathcal{Y}_n^B$. By Proposition 5.17 the ray B^{ϕ_1} lands in the dynamical plane of g_{a_0} ; this contradicts the assumption on a_0 . \square

It follows from Propositions 5.15 and 5.19 that:

Corollary 5.20. *For a parapuzzle piece \mathcal{Y} in $\mathcal{V}_1 \setminus \mathcal{W}_B$ the \mathcal{V}_2 -twin \mathcal{Y}^B exists and $a \in \text{int}(\mathcal{Y}^B)$ if and only if Y^B exists in the dynamical plane of g_a and $a \in \text{int}(Y^B)$.*

5.7. Triviality of fibers at Misiurewicz parameters.

Lemma 5.21. *If $a \in \mathcal{M}_2$ is a non- \mathfrak{B} Misiurewicz parameter, then there is a sequence \mathcal{Y}_n^B of parameter bubble pieces such that $a \in \text{int}(\mathcal{Y}_n^B)$ and $\cap \mathcal{Y}_n^B = \{a\}$.*

If $a \in \mathcal{M}_2$ is a \mathfrak{B} Misiurewicz parameter, then there are sequences \mathcal{Y}_n^B and $\mathcal{Y}_n'^B$ of parameter bubble pieces and two components $\mathcal{E}, \mathcal{E}'$ of $\mathcal{V}_2 \setminus \mathcal{M}_2$ such that $a \in \text{int}(\mathcal{Y}_n^B \cup \mathcal{Y}_n'^B \cup \mathcal{E} \cup \mathcal{E}')$ and $\cap \mathcal{Y}_n^B = \cap \mathcal{Y}_n'^B = \{a\}$.

Proof. Assume that g_a is a non- \mathfrak{B} Misiurewicz parameters; we write $a = m(c)$ as in Definition 5.11.

By definition, in the dynamical plane of f_c the critical value c maps after k iterations to a periodic point, say p of period q . Let us choose a puzzle piece Y_0 containing p and not containing any other point of the post-critical set of f_c . Then $f_c^{k+nq}(c) = p \in Y_0$; define $Y_n \ni c$ to be the pullback of Y_0 under f_c^{k+nq} along the orbit of c . We get $c \in \text{int}(\mathcal{Y}_n)$ and $\cap_n \mathcal{Y}_n = \{c\}$ by triviality of fibers for Misiurewicz parameters in \mathcal{V}_1 .

By Corollary 5.20 the pieces $\mathcal{Y}_n^B, Y_n^B, Y_0^B$ exist and $a \in \text{int}(\mathcal{Y}_n^B)$. We need to show that $\cap_n \mathcal{Y}_n^B = \{a\}$.

Assume that a' is in $\cap_n \mathcal{Y}_n^B$. Then in the dynamical plane of $g_{a'}$ the critical value $-a'$ is in Y_n^B . Moreover, $g_{a'}^{k+nq}(-a') \in Y_0^B$ for all n . We claim that $g_{a'}^{k+nq}(-a') = g_{a'}^{k+(n-1)q}(-a') = p^B$ is a periodic point. Indeed, $g_{a'}^q : g_{a'}^k(Y_{n+1}^B) \rightarrow g_{a'}^k(Y_n^B)$ has degree 1 because it is so for $f_c^q : f_c^k(Y_{n+1}) \rightarrow f_c^k(Y_n)$. Further, $g_{a'}^k(Y_{n+1}^B) \subset g_{a'}^k(Y_n^B) \subset g_{a'}^k(Y_0^B)$, so there is exactly one (periodic) point, call it p^B , that does not escape from Y_{n+1}^B under iteration of $g_{a'}^q$. Therefore, $a' = a$ as there is only one Misiurewicz parameter with this property (Thurston's rigidity theorem).

If $a \in \mathcal{V}_2$ is a \mathfrak{B} Misiurewicz parameter, then there are two \mathfrak{B} Misiurewicz parameters c, c' in \mathcal{V}_1 such that $a = m(c) = m(c')$. Let \mathcal{Y}_n (resp. \mathcal{Y}'_n) be a sequence parapuzzle pieces around c (resp. around c') as above. Then bubble parapuzzle pieces $\mathcal{Y}_n^B, \mathcal{Y}'_n^B$ contain a on the boundaries. The same argument as in the non- \mathfrak{B} case shows that if $a' \in \cap \mathcal{Y}_n^B$ or $a' \in \cap \mathcal{Y}'_n^B$, then $a' = a$. Note that a is on the boundary of exactly two components of $\mathcal{V}_2 \setminus \mathcal{M}_2$, denoted by $\mathcal{E}, \mathcal{E}'$. It is clear that $a \in \text{int}(\mathcal{Y}_n^B \cup \mathcal{Y}'_n^B \cup \mathcal{E} \cup \mathcal{E}')$. \square

5.8. Proof of Theorem 5.1. We have showed that if $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ land together at c and $\phi_1 \neq \phi_2$, then the rays $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$ land together at $m(c)$.

Assume now that a non- \mathfrak{B} ray \mathcal{R}^ϕ lands alone at γ' . Then, by Lemma 5.21, the ray \mathcal{B}^ϕ lands. We need to show that \mathcal{B}^ϕ lands alone at $m(\gamma') = f_{\gamma'} \cup_{\mathbb{S}^1} f_B$.

By Corollary 5.20 the parameter γ' is in a parapuzzle piece $\mathcal{Y} \subset \mathcal{V}_1 \setminus \mathcal{W}_B$ if and only if a is in \mathcal{Y}^B . By Lemma 5.21 parapuzzle pieces around the parameter a shrink to $\{a\}$. Combining with Lemma 5.18 we get $a = m(\gamma')$.

Suppose that a is also the landing point of a rational ray \mathcal{B}^ψ . Denote by γ'' the landing points of \mathcal{R}^ψ respectively. The same argument as above shows that $a = m(\gamma'')$. Then $\gamma' = \gamma''$ because m is injective on non- \mathfrak{B} Misiurewicz parameters.

We have shown that all parameter non- \mathfrak{B} rational rays land. By Lemma 5.6 the dynamic ray $\overline{B^\phi}$ is stable unless the parameter is on $\cup_n \overline{\mathcal{B}^{2^n \phi}}$. We conclude that B^ϕ lands unless $a \in B^{2^n \phi}$ as it is so for at least one parameter in every component of $\mathcal{V}_2 \setminus \cup_n \overline{\mathcal{B}^{2^n \phi}}$.

5.9. The map $c \rightarrow m(c)$. Recall that the map $c \rightarrow m(c)$ in Definition 5.11 is defined for the parameters on the closures of hyperbolic components and for Misiurewicz parameters.

Proposition 5.22. *The equality $m(c) = m(c')$ holds if and only if $c = c'$ or c, c' are \mathfrak{B} Misiurewicz parameters such that the external angles of c, c' are identified by L_B .*

Proof. The Misiurewicz case is well known. The statement is obvious if c or c' is in a hyperbolic component. Further, if c is parabolic, resp. indifferent (Siegel or Cremer parameter), then so is $m(c)$. Therefore, if c is indifferent and c' is parabolic, then $m(c) \neq m(c')$.

The parabolic case, i.e. c, c' are parabolic parameters, follows from Proposition 5.10. Assume c, c' are indifferent parameters. Then there are rays R^ϕ, R^ψ , and a hyperbolic component \mathcal{H} such that R^ϕ, R^ψ land at $\overline{\mathcal{H}}$ and $\overline{R^\phi} \cup \overline{R^\psi} \cup \mathcal{H}$ separate c, c' . By Theorem 5.1 the set $\overline{B^\phi} \cup \overline{B^\psi} \cup \mathcal{H}^B$ separate $m(c)$ and $m(c')$. \square

6. PROOF OF THEOREM 1.2

6.1. Small copies of the Mandelbrot set in \mathcal{V}_2 . Let us now describe small copies of the Mandelbrot set in \mathcal{V}_2 . Let $\mathcal{M}_\mathcal{H}$ be a small copy in $\mathcal{M} \setminus \mathcal{W}_B$. Consider the first renormalization map $f_c^p : \mathcal{Y}_\mathcal{H}^p \rightarrow \mathcal{W}_\mathcal{H}^0$. By Theorem 5.1 the pieces $\mathcal{Y}_\mathcal{H}^B, \mathcal{W}_\mathcal{H}^B$ exists and for $a \in \text{int}(\mathcal{W}_\mathcal{H}^B)$; we have $g_a^p : \mathcal{Y}_\mathcal{H}^B \rightarrow \mathcal{W}_\mathcal{H}^B$.

Denote by $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$ the rays that form $\partial \mathcal{W}_\mathcal{H}^B$; we assume that $0 < \phi_1 < \phi_2 < 0$ is the cyclic order. As ϕ_1, ϕ_2 are periodic there are two possibilities for \mathcal{B}^{ϕ_1} (and for \mathcal{B}^{ϕ_2}) either it is an infinite ray in bubbles or a ray within \mathcal{E}_∞ .

Lemma 6.1. *For the map $g_a^p : \mathcal{Y}_\mathcal{H}^B \rightarrow \mathcal{W}_\mathcal{H}^B$ as above the ray \mathcal{B}^{ϕ_1} (resp. \mathcal{B}^{ϕ_2}) is infinite if and only if $\mathcal{M}_\mathcal{H}$ is bounded by \mathcal{L}_B from the right (resp. from the left).*

Proof. The ray $\rho(B^{\phi_1})$ (resp. $\rho(B^{\phi_2})$) in the dynamical plane of the Basilica is finite if and only if $R^{-\phi_1}$ (resp. $R^{-\phi_2}$) land at the closure of the (periodic) Fatou component F_0 containing 0 (note that $\rho(B^{\phi_1})$ and $\rho(B^{\phi_2})$ can not land at the Fatou component containing -1 as $\mathcal{H} \not\subset \mathcal{W}_B$). This is equivalent to the landing of R^{ϕ_1} (resp. R^{ϕ_2}) at $\overline{F_0}$. The last is equivalent to the non-boundedness of $\mathcal{M}_{\mathcal{H}}$ from the right (resp. from the left). \square

It follows from Proposition 3.3 that at least one of \mathcal{B}^{ϕ_1} , \mathcal{B}^{ϕ_2} is an infinite ray and if B^{ϕ_1} or B^{ϕ_2} is a finite ray, then $\mathcal{M}_{\mathcal{H}}$ is a maximal primitive copy.

Definition 6.2 (Small copies: the first case). *If $\mathcal{M}_{\mathcal{H}}$ is bounded by \mathcal{L}_B from both sides, then the \mathcal{V}_2 -twin $\mathcal{M}_{\mathcal{H}}^B$ of $\mathcal{M}_{\mathcal{H}}$ is the closure of the set of parameters $a \in \text{int}(\mathcal{Y}_{\mathcal{H}}^B)$ such that in the dynamical plane the critical value $-a$ does not escape from $Y_{\mathcal{H}}^B$ under iteration of $g_a^p : Y_{\mathcal{H}}^B \rightarrow W_{\mathcal{H}}^B$.*

As in the quadratic case perturbing the map $g_a^p : Y_{\mathcal{H}}^B \rightarrow W_{\mathcal{H}}^B$ (thickening $Y_{\mathcal{H}}^B, W_{\mathcal{H}}^B$) we always can obtain a quadratic-like map. Therefore, by the straightening theorem $\mathcal{M}_{\mathcal{H}}^B$ is canonically homeomorphic to \mathcal{M} as well as to $\mathcal{M}_{\mathcal{H}}$.

We will refer to $g_a^p : Y_{\mathcal{H}}^B \rightarrow W_{\mathcal{H}}^B$ as *the first renormalization map*. As in \mathcal{V}_1 for $a \in \text{int}(\mathcal{Y}_{(n-1)\mathcal{H}}^B)$ we define $Y_{n\mathcal{H}}^B$ to be the preimage of $Y_{(n-1)\mathcal{H}}^B$ under $g_a^p : Y_{\mathcal{H}}^B \rightarrow W_{\mathcal{H}}^B$; we will refer to $g_a^p : Y_{n\mathcal{H}}^B \rightarrow Y_{(n-1)\mathcal{H}}^B$ as the n -th renormalization map. Let us note that if $\mathcal{M}_{\mathcal{H}}$ is bounded by \mathcal{L}_B from both sides, then $Y_{\mathcal{H}}^B$ (and so $Y_{n\mathcal{H}}^B$ for $n \geq 1$) does not intersect the periodic bubble E_{∞} . However, if $\mathcal{M}_{\mathcal{H}}$ is bounded by \mathcal{L}_B from only one side, then all $\mathcal{Y}_{(n\mathcal{H})}^B$ have ∞ on the boundaries. We modify the map $g_a^p : Y_{\mathcal{H}}^B \rightarrow W_{\mathcal{H}}^B$ as follows. Let $\dot{W}_{\mathcal{H}}^B$ be $W_{\mathcal{H}}^B$ without an open neighborhood of ∞ bounded by an internal equipotential in E_{∞} and let $\dot{Y}_{\mathcal{H}}^B$ be the pre-image of $\dot{W}_{\mathcal{H}}^B$ under $g_a^p : Y_{\mathcal{H}}^B \rightarrow W_{\mathcal{H}}^B$ for $a \in \text{int}(\dot{W}_{\mathcal{H}}^B)$.

Definition 6.3 (Small copies: the second case). *If $\mathcal{M}_{\mathcal{H}}$ is bounded by \mathcal{L}_B from exactly one side, then the \mathcal{V}_2 -twin $\mathcal{M}_{\mathcal{H}}^B$ of $\mathcal{M}_{\mathcal{H}}$ is the closure of the set of parameters $a \in \text{int}(\dot{\mathcal{Y}}_{\mathcal{H}}^B)$ such that in the dynamical plane the critical value $-a$ does not escape from $\dot{Y}_{\mathcal{H}}^B$ under iteration of $g_a^p : \dot{Y}_{\mathcal{H}}^B \rightarrow \dot{W}_{\mathcal{H}}^B$.*

Again, the straightening theorem implies that $\mathcal{M}_{\mathcal{H}}^B$ is canonically homeomorphic to \mathcal{M} as well as to $\mathcal{M}_{\mathcal{H}}$. We will refer to $g_a^p : \dot{Y}_{\mathcal{H}}^B \rightarrow \dot{W}_{\mathcal{H}}^B$ as *the first renormalization map* (of $\mathcal{M}_{\mathcal{H}}^B$).

If a parameter $a \in \mathcal{M}_2$ is not in any $\mathcal{M}_{\mathcal{H}}^B$ and is not in the closure of the main hyperbolic component of \mathcal{M}_2 , then the critical tableau used in [AY] is not periodic, i.e. g_a is not renormalizable, and by [AY, Theorem 9.1] the intersection of parapuzzle pieces containing the parameter a is $\{a\}$. In other words, triviality of the fiber holds at a .

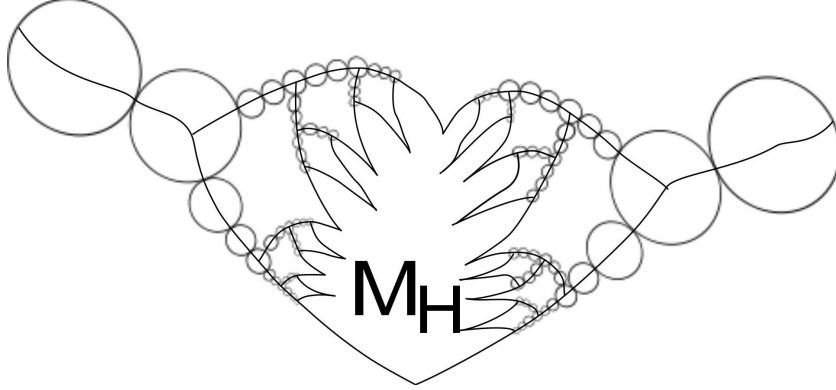


FIGURE 8. A decoration tiling in \mathcal{V}_2 (compare to Figure 5).

Definition 6.4 (The \mathcal{V}_2 -twins of renormalizable parameters). *The \mathcal{V}_2 -twin of $c \in \mathcal{M}_{\mathcal{H}}$ is $m(c) = \chi_{\mathcal{H}}(c)$, where $\chi_{\mathcal{H}} : \mathcal{M}_{\mathcal{H}} \rightarrow \mathcal{M}_{\mathcal{H}}^B$ is the canonical homeomorphism.*

For a non-parabolic non-Misiurewicz parameter $a \in \mathcal{M}_{\mathcal{H}}^B$ define T to be the set of pre-images of $\overline{\mathcal{W}_{\mathcal{H}}^B \setminus \mathcal{Y}_{\mathcal{H}}^B}$ (resp. of $\overline{\dot{\mathcal{W}}_{\mathcal{H}}^B \setminus \dot{\mathcal{Y}}_{\mathcal{H}}^B}$) under iteration of the first renormalization map. The *decoration tiling* \mathcal{T} of $\mathcal{M}_{\mathcal{H}}^B$ is the parameter counterpart to T ; see Figure 8.

Proposition 6.5 (The decoration theorem in \mathcal{V}_2). *For a small copy $\mathcal{M}_{\mathcal{H}}^B \subset \mathcal{M}_2$ of the Mandelbrot set the diameter of pieces in the decoration tiling \mathcal{T} of $\mathcal{M}_{\mathcal{H}}$ tends to 0.*

Proof. We will consider the case when $\mathcal{W}_{\mathcal{H}}^B$ is bounded by infinite rays; if one of two rays that bound $\mathcal{W}_{\mathcal{H}}^B$ is finite, then one need to modify Figure 9 accordingly; all other arguments are the same.

Let $g_a^p : Y_{2\mathcal{H}}^B \rightarrow Y_{\mathcal{H}}^B$ be the second renormalization map of $\mathcal{M}_{\mathcal{H}}^B$. It is sufficient to prove the proposition for pieces in $\mathcal{Y}_{\mathcal{H}}^B$.

Let us construct a tubing for $g_a^p : Y_{2\mathcal{H}}^B \rightarrow Y_{\mathcal{H}}^B$, where $a \in \text{int}(Y_{2\mathcal{H}}^B)$; see Figure 9 and its caption.

Denote by B^{ϕ_1}, B^{ϕ_2} the periodic and by B^{ψ_1}, B^{ψ_2} the pre-periodic rays that bounds $Y_{\mathcal{H}}^B$; we assume that $0 < \phi_1 < \psi_1 < \psi_2 < \phi_2 < 0$ is the cyclic order. As $B^{\phi_1}, B^{\phi_2}, B^{\psi_1}, B^{\psi_2}$ are infinite, the rays in the pair B^{ϕ_1}, B^{ψ_1} as well as in the pair B^{ϕ_2}, B^{ψ_2} are coincide till $Y_{\mathcal{H}}^B$. Choose $a_0 \in \text{int}(\mathcal{Y}_{2\mathcal{H}}^B)$ and choose in the dynamical plane of g_{a_0} points $x \in B^{\phi_1} \cap B^{\psi_1}$, $y \in B^{\phi_2} \cap B^{\psi_2}$ that are sufficiently close to $Y_{\mathcal{H}}^B$.

For an $a \in \text{int} \mathcal{Y}_{\mathcal{H}}^B$ define $x(a), y(a)$ to be the images of $x(a_0), y(a_0)$ under $h_a^{-1} \circ h_{a_0}$.

In the dynamical plane of g_a choose a smooth Jordan curve $\gamma(a)$ such that:

- $\gamma(a)$ surrounds $Y_{\mathcal{H}}^B$ and is sufficiently close to $\partial Y_{\mathcal{H}}^B$;
- $\gamma(a) \cap (B^{\phi_1} \cap B^{\psi_1}) = \{x\}$ and $\gamma(a) \cap (B^{\phi_2} \cap B^{\psi_2}) = \{y\}$;

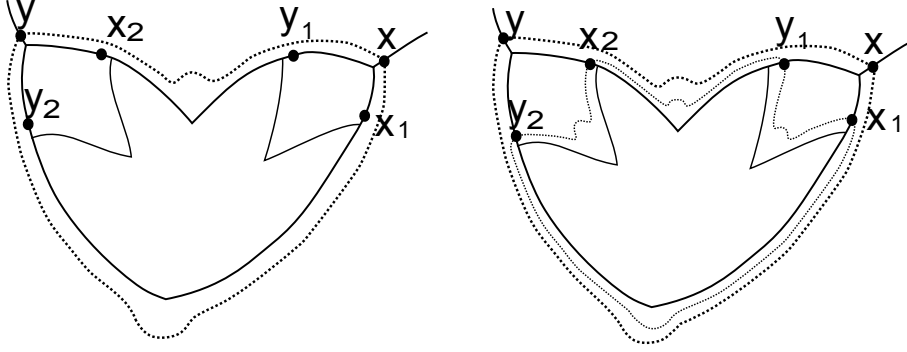


FIGURE 9. Illustration to the construction of a tubing in the proof of Proposition 6.5. The copy $\mathcal{M}_{\mathcal{H}}^B$ is encoded by the second renormalization map $g_a^p : Y_{2\mathcal{H}}^B \rightarrow Y_{\mathcal{H}}^B$. The rays in bubbles that bound $Y_{2\mathcal{H}}^B, Y_{\mathcal{H}}^B$ are drawn. Left: a curve $\gamma = \gamma(a)$ surrounds $Y_{\mathcal{H}}^B$ and intersects the rays in bubbles at x and y . Right: the pre-image of γ under $g_a^p : Y_{2\mathcal{H}}^B \rightarrow Y_{\mathcal{H}}^B$ surrounds $Y_{2\mathcal{H}}^B$ and intersects the rays in bubbles at x_1, x_2, y_1, y_2 , where $g_a^p(x_1) = g_a^p(x_2) = x$ and $g_a^p(y_1) = g_a^p(y_2) = y$. We defined $T_a(Q_R)$ to be the closed annulus bounded by $\gamma(a)$ and $\gamma_2(a)$.

- the pre-image $\gamma_2(a)$ of $\gamma(a)$ along $g_a^p : Y_{\mathcal{H}}^B \rightarrow Y_{2\mathcal{H}}^B$ is inside the open disc surrounded by $\gamma(a)$ and containing $Y_{\mathcal{H}}^B$; and
- $\gamma(a)$ depends smoothly on a .

Choose a smooth embedding T of $Q_R \times \text{int}(\mathcal{Y}_{\mathcal{H}}^B)$, where $R > 1$, into $\mathbb{C} \times \text{int}(\mathcal{Y}_{\mathcal{H}}^B)$ over $\text{int}(\mathcal{Y}_{\mathcal{H}}^B)$ such that:

- $T_a(Q_R)$, is the closed annulus bounded by $\gamma(a)$ and $\gamma_2(a)$;
- for all z in

$$(11) \quad T_a(Q_R) \cap (B^{\phi_1} \cup B^{\psi_1} \cup B^{\phi_2} \cup B^{\psi_2})$$

we have

$$(12) \quad T_{a'} \circ T_{a''}^{-1} = h_{a'}^{-1} \circ h_{a''};$$

i.e. T_a and h_a agree on (11).

By the straightening theorem we get the map

$$\chi = \chi(T) : \text{int}(\mathcal{Y}_{2\mathcal{H}}^B) \rightarrow \mathcal{V}_1.$$

We will now show that $\chi(\mathcal{T})$ is a generalized decoration tiling of $\chi(\mathcal{M}_{\mathcal{H}}^B) = \mathcal{M}$. Note that χ may not be a homeomorphism, but it is a homeomorphism on $\mathcal{M}_{\mathcal{H}'}^B$ and on $\partial\mathcal{Y}_{n\mathcal{H}'}^B$ as we will show.

Define:

$$\mathcal{X}^1 = \chi(\mathcal{Y}_{\mathcal{H}}^B), \quad \mathcal{X}^2 = \chi(\mathcal{Y}_{2\mathcal{H}}^B),$$

$$X^1(\chi(a)) = \chi(a)(Y_{\mathcal{H}}^B(a)), \quad X^2(\chi(a)) = \chi(a)(Y_{2\mathcal{H}}^B(a)).$$

We need to show that $f_c : X^1(c) \rightarrow X^2(c)$ for $c = \chi(a) \in \text{int}(\mathcal{X}^2)$, $X^1(c)$, $X^2(c)$ are generalized puzzle pieces, and $X^1(c)$, $X^2(c)$ are the dynamical counterparts of \mathcal{X}^1 , \mathcal{X}^2 .

The first claim follows directly from the construction of the straightening map because g_a^p and $f_{\chi(a)}$ are quasiconformally conjugate in neighborhoods of their filled in Julia sets, Subsection 2.4. This also implies that the forward orbit of $\partial X^1(c)$, $\partial X^2(c)$ does not intersect the interiors of $X^1(c)$, $X^2(c)$. Because $X^1(c)$, $X^2(c)$ contain the critical value, markings for $X^1(c)$, $X^2(c)$ are irrelevant. Let us now show that $X^1(c')$, $X^2(c')$ are combinatorially equivalent to $X^1(c'')$, $X^2(c'')$.

By (12) and (5) we have:

$$\chi(a') \circ (h_{a'}^{-1} \circ h_{a''}) = \chi(a') \circ T_{a'} \circ T_{a''}^{-1} = B_{\chi(a')}^{-1} \circ B_{\chi(a'')} \circ \chi(a'')$$

or

$$(13) \quad \chi(a') \circ (h_{a'}^{-1} \circ h_{a''}) = B_{\chi(a')}^{-1} \circ B_{\chi(a'')} \circ \chi(a'')$$

on $(B^{\phi_1} \cup B^{\phi_2} \cup B^{\psi_1} \cup B^{\psi_2}) \cap T_{a''}(Q_R)$. As $g_a^p : \partial \mathcal{Y}_{2\mathcal{H}}^B \rightarrow \partial \mathcal{Y}_{\mathcal{H}}^B$ is a covering and as all points in

$$\partial Y_{\mathcal{H}}^B \cup \partial Y_{2\mathcal{H}}^B \setminus \{\text{the landing points of rays}\}$$

are preimages of $(B^{\phi_1} \cup B^{\phi_2} \cup B^{\psi_1} \cup B^{\psi_2}) \cap T_{a''}(Q_R)$, taking pullbacks of (13) under iteration of $g_a^p : \partial \mathcal{Y}_{2\mathcal{H}}^B \rightarrow \partial \mathcal{Y}_{\mathcal{H}}^B$ we obtain that (13) holds on $\partial Y_{\mathcal{H}}^B(a'') \cup \partial Y_{2\mathcal{H}}^B(a'') \setminus J_{a''}$. Therefore, $B_{\chi(a')}^{-1} \circ B_{\chi(a'')}$ maps

$$\partial X^1(\chi(a'')) \cup \partial X^2(\chi(a'')) = \chi(a'') (\partial Y_{\mathcal{H}}^B(a'') \cup \partial Y_{2\mathcal{H}}^B(a''))$$

homeomorphically to $\partial X^1(\chi(a')) \cup \partial X^2(\chi(a'))$ except at finitely many points in the Julia sets that are the landing points of rays.

This shows that $X_1(c'')$, $X_2(c'')$ are combinatorially equivalent to $X_1(c')$, $X_2(c')$. Similarly, \mathcal{X}_1 , \mathcal{X}_2 are the parameter counterparts to $X_1(c'')$, $X_2(c'')$. We conclude that $\chi(\mathcal{T})$ is a generalized decoration tiling.

Assume now that the diameter of pieces in \mathcal{T} does not tend to 0. Then there is an infinite sequence $\mathcal{T}_1, \mathcal{T}_2, \dots$ of pieces in \mathcal{T} and there are points $a_1 \neq a_2$ such that a_1, a_2 are the limiting points of $\{T_i\}$. We claim that $a_1, a_2 \in \mathcal{M}_{\mathcal{H}}^B$. Indeed, if, say, $a_1 \notin \mathcal{M}_{\mathcal{H}}^B$, then in the dynamical plane of g_{a_1} the critical value $-a_1$ escapes from $Y_{2\mathcal{H}}^B$ under iteration of $g_{a_1}^p : Y_{2\mathcal{H}}^B \rightarrow Y_{\mathcal{H}}^B$, say in N iterations. Then for all a close to a_1 the critical value $-a$ escapes from $Y_{2\mathcal{H}}^B$ under iteration of $g_a^p : Y_{2\mathcal{H}}^B \rightarrow Y_{\mathcal{H}}^B$ in less or equal than $N+2$ iterations. Observe that there are only finitely many pieces in \mathcal{T} with bounded by $N+2$ escaping time. Whence a_1 (and a_2) is in $\mathcal{M}_{\mathcal{H}}^B$.

As χ is a homeomorphism on $\mathcal{M}_{\mathcal{H}}^B$, it follows that $\chi(a_1) \neq \chi(a_2)$. As χ is continuous the sequence $\{\chi(T_i)\}$ has two limiting points $\chi(a_1)$ and $\chi(a_2)$. This contradicts Theorem 2.7 that says that the diameter of pieces in \mathcal{T} tends to 0. \square

6.2. No ghost limbs in \mathcal{V}_2 . No ghost limb theorem in \mathcal{V}_1 says that if a parameter c is in $\mathcal{W}_{\mathcal{H}} \cap \mathcal{M}$, then either $c \in \overline{\mathcal{H}}$ or c is in a limb attached to $\overline{\mathcal{H}}$.

The next proposition is a \mathcal{V}_2 -analogy of no ghost limb theorem in \mathcal{V}_1 ; for our convenience, we say that $\mathcal{W}_{\mathcal{H}}$ is \mathcal{V}_2 if \mathcal{H} is the main hyperbolic component of \mathcal{M}_2 .

Proposition 6.6. *If $a \in \text{int}(\mathcal{W}_{\mathcal{H}}^B)$, then either $a \in \overline{\mathcal{H}^B}$ or $a \in \mathcal{W}_{\mathcal{H}'}^B$ for a component $H'^B \in \mathcal{W}_{\mathcal{H}}^B$ attached to \mathcal{H}^B .*

Proof. Consider a periodic cycle $\bar{p} = \{p_1, \dots, p_n\}$ associated with \mathcal{H}^B . If $a \in \text{int}(\mathcal{W}_{\mathcal{H}}^B)$ but $a \notin \overline{\mathcal{H}^B}$, then \bar{p} is a repelling cycle and either there are periodic rays in bubbles landing at \bar{p} or a is in a parameter ray in bubbles with a periodic angle.

If a is in a parameter ray, then the statement is obvious.

If there are periodic rays landing at \bar{p} , then choose two rays B^{ϕ_1}, B^{ϕ_2} landing at \bar{p} and separating $-a$ from all other rays landing at \bar{p} . Then a is in the interior of the parapuzzle piece $\mathcal{Y}^B(\phi_1, \phi_2)$ bounded by $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$. It follows from Proposition 5.22 that $\mathcal{Y}^B(\phi_1, \phi_2)$ is $\mathcal{W}_{\mathcal{H}'}^B$ for some $\mathcal{H}'^B \in \mathcal{W}_{\mathcal{H}}^B$ attached to \mathcal{H}^B . \square

6.3. Triviality of fibers for hyperbolic components. Recall that by a *combinatorial disc* in \mathcal{V}_2 we mean a closed topological disc \mathcal{D} such that $\partial\mathcal{D}$ is in a finite union of parameter rational rays in bubbles and simple curves in the closures of hyperbolic components. A sequence a_n tends to a_∞ *combinatorially* if for every combinatorial disc \mathcal{D} such that $a_\infty \in \text{int}(\mathcal{D})$ all but finitely many a_n are in \mathcal{D} .

Later we will prove Yoccoz's results for \mathcal{V}_2 which say that a sequence a_n tends combinatorially to a_∞ if and only if a_n tends to a_∞ in the usual topology (Theorem 6.8).

Proposition 6.7. *Theorem 6.8 holds for the closures of hyperbolic components in \mathcal{M}_2 .*

Proof. We will use ideas from [Sc1]. The claim is obvious for the interiors of hyperbolic components.

Let a be a parameter on $\partial\mathcal{H}^B \subset \mathcal{M}_2$. We will construct a sequence $\{\mathcal{F}_n^B\}$ of nested combinatorial topological discs such that $a \in \text{int}(\mathcal{F}_n)$ and $\cap_n \mathcal{F}_n = a$. This will imply the claim of the proposition.

Indifferent case. Suppose that a is an indifferent (non-parabolic) point on $\partial\mathcal{H}$. Choose monotone sequences $\{b_n\}, \{c_n\}$ on $\partial\mathcal{H}$ (recall that $\partial\mathcal{H}$ is a Jordan curve) such that b_n, c_n are parabolic points and $\{b_n\}, \{c_n\}$ converge from different sides to a . Choose periodic rays $\mathcal{B}^{\phi_n}, \mathcal{B}^{\psi_n}$ landing at b_n, c_n respectively and a simple curve γ_n in $\overline{\mathcal{H}^B}$ connecting b_n, c_n such that γ_n tends to a , does not contain a , and γ_{n-1} and a are in different components of $\overline{\mathcal{H}^B} \setminus \gamma_n$; i.e. γ_n is a monotone sequence. We define \mathcal{F}_n to be the closed topological disc bounded by $\mathcal{B}^{\phi_n} \cup \mathcal{B}^{\psi_n} \cup \gamma_n$ and containing a . By definition, \mathcal{F}_n is a

combinatorial neighborhood of a . It follows from the construction that $(\cap_n \mathcal{F}_n) \cap \overline{\mathcal{H}^B} = a$.

Assume that $a' \in \cap_n \mathcal{F}_n$. We claim that $a' = a$. Indeed, a' can not be in $\mathcal{W}_{\mathcal{H}'}^B$ for a component $H'^B \in \mathcal{W}_{\mathcal{H}}^B$ attached to \mathcal{H}^B because $\mathcal{W}_{\mathcal{H}'}^B$ intersects only finitely many sets in \mathcal{F}_n . By Proposition 6.6 we have $a' \in \mathcal{H}^B$ and so $a = a'$.

Parabolic non-primitive case. Suppose that $a \in \overline{\mathcal{H}^B} \cap \overline{\mathcal{H}'^B}$, where \mathcal{H}'^B is a hyperbolic component attached to \mathcal{H}^B . Choose four monotone sequences $\{b_n\}, \{c_n\}, \{d_n\}, \{f_n\}$ on $\partial\mathcal{H}^B \cup \partial\mathcal{H}'^B$ consisting of parabolic points and converging to a from different sides of $\partial\mathcal{H}^B \cup \partial\mathcal{H}'^B$. Choose simple curves β_n, γ_n in $\overline{\mathcal{H}^B}, \overline{\mathcal{H}'^B}$ respectively connecting pairs of points in b_n, c_n, d_n, f_n such that β_n, γ_n converge to a , do not contain a , and $a, \beta_{n-1}, \gamma_{n-1}$ are in different components of $\overline{\mathcal{H}^B} \cup \overline{\mathcal{H}'^B} \setminus (\beta_n \cup \gamma_n)$. We define $\mathcal{F}_n \ni a$ to be the closed topological disc bounded by periodic rays landing at b_n, c_n, d_n, f_n and by β_n, γ_n . As in the indifferent case $a \in \text{int } F_n$ and $\cap F_n = \{a\}$.

Parabolic primitive case. Suppose that \mathcal{H}^B is a primitive (i.e. it is not attached to a hyperbolic component of smaller period), \mathcal{H}^B is not the main hyperbolic component of \mathcal{M}_2 , and a is the root of \mathcal{H}^B . Let $\{\mathcal{B}^{\phi_\infty}, \mathcal{B}^{\psi_\infty}\}$ be the pair of periodic rays landing at a ; we assume that $0 < \phi_\infty < \psi_\infty < 0$ is the cyclic order. Choose a sequence of pairs $\{\mathcal{B}^{\phi'_n}, \mathcal{B}^{\psi'_n}\}$ landing together rational rays such that $\phi'_n \rightarrow \phi_\infty$ and $\psi'_n \rightarrow \psi_\infty$ and $0 < \phi'_n < \phi_\infty < \psi_\infty < \psi'_n < 0$ is the cyclic order, this sequence exists by Theorem 5.1 and the well known fact that the same statement is true for \mathcal{V}_1 .

Let us remind that $\mathcal{B}^{\phi_\infty}$ or $\mathcal{B}^{\psi_\infty}$ is possibly a finite ray in \mathcal{E}_∞ . Denote by \mathcal{U}_n the open disc in \mathcal{E}_∞ bounded by the internal equipotential of height $1/2^n$; we have $\mathcal{U}_i \subset \mathcal{U}_{i+1}$ and $\mathcal{E}_\infty = \cup_n \mathcal{U}_n$. Choose sequences $\{\mathcal{B}^{\phi_n}\}, \{\mathcal{B}^{\psi_n}\}, \{\gamma_n\}$ as in the indifferent case. Define $\widetilde{\mathcal{F}}_n$ to be the closed topological disc bounded by $\mathcal{B}^{\phi_n} \cup \mathcal{B}^{\psi_n} \cup \gamma_n \cup \mathcal{B}^{\phi'_n} \cup \mathcal{B}^{\psi'_n}$ and containing a . Define $\mathcal{F}_n := \widetilde{\mathcal{F}}_n \setminus \mathcal{U}_n$.

Consider $a' \in \cap_n \text{int } \mathcal{F}_n$. If $a' \in \mathcal{W}_{\mathcal{H}}^B$, then the same argument as in the indifferent case shows that $a' = a$. Let us assume $a' \notin \mathcal{W}_{\mathcal{H}}^B$.

By construction, all except possibly \mathcal{E}_∞ components of $\mathcal{V}_2 \setminus \mathcal{M}_2$ intersect at most finitely many \mathcal{F}_n . A given point in \mathcal{E}_∞ is also in at most finitely many \mathcal{F}_n because $\mathcal{E}_\infty = \cup_n \mathcal{U}_n$. Therefore, $a' \in \partial\mathcal{M}_2$.

If a' is not in any copy of the Mandelbrot set, then by [AY, Theorem 9.1] there is a sequence \mathcal{Y}_n^B of parapuzzle pieces containing a' such that $\text{diam}(\mathcal{Y}_n^B)$ tends to 0. Hence for big n the piece \mathcal{Y}_n^B does not intersect $\overline{\mathcal{H}^B}$. Therefore, \mathcal{Y}_n^B is separated from \mathcal{H}^B by rational rays. This implies that for big m the sets \mathcal{Y}_n^B and \mathcal{F}_m do not intersect; this contradicts the definition of a' .

Assume that a' is in a small copy $\mathcal{M}_{\mathcal{H}'}^B$ of the Mandelbrot set. Then either $a' \in \mathcal{M}_{\mathcal{H}'}^B$ or $a' \notin \mathcal{M}_{\mathcal{H}'}^B$. If $a' \in \mathcal{M}_{\mathcal{H}'}^B$, then $a' = m(c')$ and

$a = m(c)$ as in Definition 5.11 and there are landing together rational rays $\mathcal{R}^{\phi_1}, \mathcal{R}^{\phi_2}$ separating c' and c . Therefore, $\mathcal{B}^{\phi_1}, \mathcal{B}^{\phi_2}$ land together and separate a' and a .

If $a' \notin \mathcal{M}_{\mathcal{H}'}^B$, then the rays $\mathcal{B}^{\phi'_n}, \mathcal{B}^{\psi'_n}$ must separate $\mathcal{M}_{\mathcal{H}'}^B$ and \mathcal{H}^B for sufficiently big n as this is true in \mathcal{V}_1 for $\mathcal{R}^{\phi'_n}, \mathcal{R}^{\psi'_n}, \mathcal{M}_{\mathcal{H}'}$, and \mathcal{H} .

The special case. Assume a is the root of the main hyperbolic component or $a = 0$. Choose sequences $\{\mathcal{B}^{\phi_n}\}, \{\mathcal{B}^{\psi_n}\}, \{\gamma_n\}$ as in the indifferent case. Let \mathcal{U}_n be the open disc in \mathcal{E}_∞ bounded by the internal equipotential of height $1/2^n$ (see the primitive parabolic case). Define $\widetilde{\mathcal{F}}_n$ to be the closed topological disc bounded by $\mathcal{B}^{\phi_n} \cup \mathcal{B}^{\psi_n} \cup \gamma_n$ and containing a . Define $\mathcal{F}_n := \widetilde{\mathcal{F}}_n \setminus \mathcal{U}_n$. The same arguments as above shows that $\cap_n \mathcal{F}_n = \{a\}$. □

6.4. Yoccoz's results in \mathcal{V}_2 .

Theorem 6.8 (Yoccoz's results in \mathcal{V}_2). *Assume a' is not infinitely renormalizable. Then a sequence a_n tends to a' combinatorially if and only if a_n tends to a' in the usual topology.*

Proof. The proof is the combination of [AY, Theorem 9.1], Propositions 6.7, 6.5, and Lemma 5.21.

The statement is obvious for points in hyperbolic components. The case when a is a Misiurewicz parameter follows from Lemma 5.21. The case when a is on the boundary of a hyperbolic component in \mathcal{M}_2 follows from Propositions 6.7. [AY, Theorem 9.1] covers the case when a is not renormalizable and is not in the closure of a hyperbolic component of \mathcal{M}_2 .

Let us verify the last case: a is in a small copy $\mathcal{M}_{\mathcal{H}}^B$ of the Mandelbrot set and a is neither a Misiurewicz parameter nor on the boundary of a hyperbolic component of \mathcal{M}_2 .

Denote by $\chi_{\mathcal{H}} : \mathcal{M}_{\mathcal{H}}^B \rightarrow \mathcal{M}_{\mathcal{H}}$ the straightening map. We will consider two cases.

Case 1. If $a_n \in \mathcal{M}_{\mathcal{H}}^B$, then $\chi(a_n)$ tends to $\chi(a')$ combinatorially. By Yoccoz's results for \mathcal{M} we see that $\chi(a_n)$ tends to $\chi(a')$ in the usual topology. As χ is a homeomorphism, a_n tends to a' in the usual topology.

Case 2. If $a_n \notin \mathcal{M}_{\mathcal{H}}^B$, then we may assume (by removing finitely many points in a_n) that all a_n are in $\mathcal{W}_{\mathcal{H}}^B$ (resp. in $\dot{\mathcal{W}}_{\mathcal{H}}^B$ if $\mathcal{W}_{\mathcal{H}}^B$ is bounded by a finite ray). Consider the decoration tiling \mathcal{T} for $\mathcal{M}_{\mathcal{H}}^B$; denote by \mathcal{T}_n the piece in \mathcal{T} containing a_n ; choose a point b_n in $\mathcal{T}_n \cap \mathcal{M}_{\mathcal{H}}^B$.

As a' is not a Misiurewicz parameter, a piece in \mathcal{T} may occur only finitely many times in $\{\mathcal{T}_n\}$. By Proposition 6.5 the diameter of pieces in $\{\mathcal{T}_n\}$ tends to 0. Thus the distance between a_n and b_n tends to 0. As a consequence, b_n tends to a' combinatorially. By Case 1 the sequence

b_n tends to a' in the usual topology. As the distance between a_n and b_n tends to 0 the sequence a_n tends to a' in the usual topology. \square

Corollary 6.9. *The set \mathcal{M}_2 is locally connected.*

Proof. Recall that the boundary of a component of $\mathcal{V}_2 \setminus \mathcal{M}_2$ is locally connected (Theorem 4.7). We need to show that the diameter of components in $\mathcal{V}_2 \setminus \mathcal{M}_2$ tends to 0.

By Theorem 6.8 big components (with diameters at least some fixed $\varepsilon > 0$) of $\mathcal{V}_2 \setminus \mathcal{M}_2$ can not accumulate at non infinitely renormalizable parameters. But by Proposition 6.5 and the observation that a component of $\mathcal{V}_2 \setminus \mathcal{M}_2$ can intersect only three pieces in a decoration tiling (see Figure 8) it follows that big components of $\mathcal{V}_2 \setminus \mathcal{M}_2$ can not accumulate at a small copy $\mathcal{M}_{\mathcal{H}}^B$ of the Mandelbrot set. \square

6.5. Proof of Theorem 1.2. Recall that by \mathcal{L}_B we denote a lamination of the Basilica embedded into \mathcal{V}_1 , by $\pi : \mathcal{V}_1 \rightarrow \mathcal{V}_2'$ the quotient map, by $\mathcal{M}'_2 = \pi(\mathcal{M})$ the image of \mathcal{M} in \mathcal{V}_2' , and $\widehat{\mathcal{M}} = \mathcal{M} \cup \mathcal{L}_B \cup \mathcal{W}_B = \pi^{-1}(\mathcal{M}'_2)$.

By *small copies* of \mathcal{M} in \mathcal{M}'_2 we mean the images of small copies of $\mathcal{M} \setminus \mathcal{W}_B$ under π . It follows that small copies of \mathcal{M} in \mathcal{M}'_2 are canonically homeomorphic to \mathcal{M} .

Step 1. Let us construct a bijection ω between \mathcal{M}'_2 and \mathcal{M}_2 .

If $c \in \overline{\mathcal{H}} \subset (\mathcal{M} \setminus \mathcal{W}_B)$ or $c \in \mathcal{M} \setminus \mathcal{W}_B$ is a Misiurewicz parameter, then define $\omega(\pi(c)) := m(c)$ as in Definition 5.11.

If c is in a small copy $\mathcal{M}_{\mathcal{H}}$, then we define $\omega(\pi(c)) := \chi_{\mathcal{H}}(c)$, where $\chi_{\mathcal{H}} : \mathcal{M}_{\mathcal{H}} \rightarrow \mathcal{M}_{\mathcal{H}}^B$ is the canonical homeomorphism.

If $c \in \mathcal{V}_1 \setminus \mathcal{W}_B$ is not in the closure of a hyperbolic component, is not a Misiurewicz parameter, and is not an infinitely renormalizable parameter, then define $\omega(\pi(c))$ to be the unique point in $\cap_{\mathcal{Y}} \mathcal{Y}^B$, where the intersection is taken over all puzzle pieces containing c .

All above cases are compatible; this defines the bijection $\omega : \mathcal{M}'_2 \rightarrow \mathcal{M}_2$. Moreover, ω preserves the combinatorics:

$$(14) \quad \omega(\pi(\mathcal{Y}_t \cap \mathcal{M})) = \mathcal{Y}_t^B \cap \mathcal{M}_2$$

for a parapuzzle piece \mathcal{Y}_t in $\mathcal{V}_1 \setminus \mathcal{W}_B$.

Step 2. The bijection $\omega : \mathcal{M}'_2 \rightarrow \mathcal{M}_2$ is a homeomorphism.

We need to prove that if $c_n \in \mathcal{M}'_2$ tends to $c_\infty \in \mathcal{M}'_2$, then $\omega(c_n)$ tends to $\omega(c_\infty)$.

Case 1. Assume c_∞ belongs to at most finitely many small copies of the Mandelbrot set; then the same is true for $\omega(c_\infty)$. By (14) the sequence $\omega(c_n)$ tends to $\omega(c_\infty)$ combinatorially. By Theorem 6.8 the sequence $\omega(c_n)$ tends to $\omega(c_\infty)$ in the usual topology.

Case 2. Assume c_∞ and all c_n belong to a single small copy $\pi(\mathcal{M}_\mathcal{H})$. Then the statement follows from the definition of ω because ω coincides with the canonical homeomorphism from $\pi(\mathcal{M}_\mathcal{H})$ to $\omega(\pi(\mathcal{M}_\mathcal{H})) = \mathcal{M}_\mathcal{H}^B$ and π restricted to $\mathcal{M}_\mathcal{H}$ is a homeomorphism.

Case 3. Assume c_∞ belongs to infinitely many copies of the Mandelbrot set (i.e., c_∞ is infinitely renormalizable), $\mathcal{M}_\mathcal{H} \subset \mathcal{M} \setminus \mathcal{W}_B$ is a small copy containing $\pi^{-1}(c_\infty)$, and assume that all $\pi^{-1}(c_n)$ do not intersect $\mathcal{M}_\mathcal{H}$. Recall that $\pi^{-1}(c_n)$ is either a single point or a pair of \mathfrak{B} Misiurewicz points connected by a leaf of \mathcal{L}_B .

Let \mathcal{T}_n be a component of $\mathcal{M} \setminus \mathcal{M}_\mathcal{H}$ that intersects $\pi^{-1}(c_n)$. Let a_n be the intersection of $\mathcal{M}_\mathcal{H}$ and \mathcal{T}_n . As $\pi^{-1}(c_\infty)$ belongs to infinitely many copies of \mathcal{M} , it follows that $\pi^{-1}(c_\infty) \neq a_n$ for all n . Therefore, the diameter of $\pi^{-1}(c_n)$ tends to 0 (Corollary 3.5) and only finitely many $\pi^{-1}(c_n)$ intersect \mathcal{T}_k for each fixed k . Hence by the decoration theorem the distance between $\pi^{-1}(c_n)$ and a_n tends to 0, and so the sequence a_n tends to $\pi^{-1}(c_\infty)$. By Case 2 we obtain that $\omega(\pi(a_n))$ tends to $\omega(c_\infty)$.

By removing finitely many elements in the sequence c_n we may assume that \mathcal{T}_n does not contain the main hyperbolic component of \mathcal{M} for all n . Then a_n is a single Misiurewicz point. It follows from (14) that a decoration of $\pi(\mathcal{T}_n)$ is in a unique piece, say \mathcal{T}_n^B , of the decoration tiling \mathcal{T} of $\mathcal{M}_\mathcal{H}^B$. As the diameter of pieces in \mathcal{T} tends to 0 the distance between $\omega(a_n)$ and $\omega(c_n)$ tends to 0. We conclude that $\omega(c_n)$ tends to $\omega(c_\infty)$.

Step 3. We claim that the homeomorphism $\omega : \mathcal{M}'_2 \rightarrow \mathcal{M}_2$ preserves the orientation of the complement. Let us first show how to identify the components of $\mathcal{V}_1 \setminus \widehat{\mathcal{M}}$ with the components of $\mathcal{V}_2 \setminus \mathcal{M}_2$. Let \mathcal{U} be a component of $\mathcal{V}_1 \setminus \widehat{\mathcal{M}}$. Then $U := (1/z \circ \mathcal{B})(\mathcal{U})$ is a component of $\mathbb{D} \setminus L_B$. Under the quotient map $\overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}/L_B \approx K_B$ the component U is mapped to the component of $\text{int}(K_B \setminus \{1/2\text{-limb}\})$. By Theorem 4.7 the components of $\text{int}(K_B \setminus \{1/2\text{-limb}\})$ are identified with the components of $\mathcal{V}_2 \setminus \mathcal{M}_2$; denote by \mathcal{U}^B the component in $\mathcal{V}_2 \setminus \mathcal{M}_2$ corresponding to \mathcal{U} under the above identifications.

We claim that if $x \in \partial\pi(\mathcal{U})$, then $\omega(x) \in \partial\mathcal{U}^B$ and, moreover, ω preserves the orientation of \mathcal{U} . The above claims are obvious (the last part of Theorem 4.7) for the set

$$\text{Mis}(U) = \{\pi(y) : y \text{ is a } \mathfrak{B} \text{ Misiurewicz parameter on } \partial\mathcal{U}\};$$

as $\text{Mis}(U)$ is dense in $\partial\pi(\mathcal{U})$ the statement holds for all points on $\partial\pi(\mathcal{U})$.

Step 4. By Proposition 2.10 the homeomorphism ω extends to a homeomorphism from \mathcal{V}'_2 onto \mathcal{V}_2 .

7. MATINGS WITH LAMINATIONS IN THE DYNAMICAL PLANES

Consider a quadratic polynomial $z^2 + c$, where $c \notin \mathcal{W}_B$. Let L_B be a z^2 -invariant lamination of the Basilica. The composition $B_c^{-1} \circ 1/z$ embeds L_B into the dynamical plane of $f_c = z^2 + c$; denote by

$$L_B^c = B_c^{-1} \circ (1/z)(L_B) = \bigcup_{l \in L_B} \overline{(1/z \circ B_c)^{-1}(l|_{\mathbb{D}})}$$

the embedding.

It follows from Definition 2.12 that L_B^c has the following description:

$$(15) \quad L_B^c = \langle 1/3, 2/3 \rangle \cup_{n \geq 0} f_c^{-n}(\langle 5/6, 1/6 \rangle).$$

For instance, if l is a leaf of L_B^c and $l \neq \langle 1/3, 2/3 \rangle$, then pre-images of l under f_c^n are also leaves of L_B^c .

7.1. The set $\widehat{K}_c = K_c \cup L_B^c$. Define

$$\widehat{K}_c = K_c \cup L_B^c.$$

Proposition 7.1. *If $c \in \mathcal{M}_{\mathcal{H}}$ and $\mathcal{M}_{\mathcal{H}}$ is bounded by $\mathcal{L}_{\mathcal{H}}$ from both sides, then the diameter of components in $\mathbb{C} \setminus \widehat{K}_c$ tends to 0 and the diameter of leaves in L_B^c tends to 0.*

Proof. In the proof we will use the hyperbolic contraction. Let K_1, K_2, \dots, K_p be the periodic cycle of small filled in Julia sets associated with $\mathcal{M}_{\mathcal{H}}$. It follows from the boundedness condition that no point in a component of $\mathbb{C} \setminus (\widehat{K}_c)$ is in $\cup_i K_i$.

By construction, the composition $B_c^{-1} \circ (1/z)$ identifies components of $\mathbb{D} \setminus L_B$ and $\widehat{\mathbb{C}} \setminus \widehat{K}_c$. By Definition 2.12:

$$\mathbb{D} \setminus L_B = U_0 \cup U_{-1} \cup_{n \geq 0} (z^{-2^n})(U_1),$$

where U_0, U_{-1} , and U_1 are the components of $\mathbb{D} \setminus L_B$ that are mapped under the quotient map $\overline{\mathbb{D}}/L_B \approx K_B$ onto the components of $\text{int}(K_B)$ containing 0, -1, and 1 respectively.

Define V_0, V_{-1} , and V_1 to be the images of U_0, U_{-1} , and U_1 under the composition $B_c^{-1} \circ (1/z)$. We get:

$$\widehat{\mathbb{C}} \setminus \widehat{K}_c = V_0 \cup V_{-1} \cup_{n \geq 0} f_c^{-n}(V_1).$$

As the critical orbit is in $\cup_i K_i$ the map

$$f_c : \mathbb{C} \setminus f_c^{-1} \left(\bigcup_i K_i \right) \longrightarrow \mathbb{C} \setminus \bigcup_i K_i$$

is a covering of hyperbolic surfaces, and

$$\mathbb{C} \setminus f_c^{-1} \left(\bigcup_i K_i \right) \hookrightarrow \mathbb{C} \setminus \bigcup_i K_i$$

is a contracting inclusion.

As V_1 is disjoint from $\cup_i K_i \cup \{\infty\}$ we see that V_1 has a finite hyperbolic diameter in $S = \mathbb{C} \setminus \cup_i K_i$. Then the hyperbolic diameter of components in $\mathbb{C} \setminus \widehat{K}_c$ is uniformly bounded; moreover, the hyperbolic diameter of components in $\mathbb{C} \setminus \widehat{K}_c$ that are in a fixed compact subset of S tends to 0. Therefore, the Euclidean diameter of components in $\mathbb{C} \setminus \widehat{K}_c$ tends to 0.

Similarly, the Euclidean diameter of leaves in L_B^c tends to 0. \square

We say that two non-equal points are equivalent under L_B^c if these points are in a leaf of L_B^c .

Corollary 7.2. *The equivalence relation L_B^c is closed, canonical on K_c , and satisfies the condition of Moore's theorem.*

Proof. Follows from Proposition 7.1 and the observation that leaves in L_B^c land at different points (\mathfrak{B} rays land alone as $c \notin \mathcal{W}_B$). \square

By Moore's theorem $\widehat{\mathbb{C}}/L_B^c$ is topologically a sphere \mathbb{S}^2 . Denote by π_c the quotient map $\widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}/L_B^c \approx \mathbb{S}^2$ and by $K_{m(c)}'^B = \pi_c(K_c)$ the image of K_c in \mathbb{S}^2 . We have $\pi_c^{-1}(K_{m(c)}'^B) = \widehat{K}_c$.

7.2. Combinatorial convergence in dynamical planes. Recall that by $m(c)$ we denote the image of c in $\mathcal{V}_2 = \mathcal{M} \cup_{\mathbb{S}^1} L_B$.

By a *combinatorial disc* in the dynamical plane of $g_a : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ we mean a closed topological disc D such that ∂D is in a finite union of rational rays in bubbles and simple curves in the closures of attracting Fatou components.

By a *combinatorial disc* in the dynamical plane of $f_c : \mathbb{C} \rightarrow \mathbb{C}$ we mean a closed topological disc D such that ∂D is in a finite union of rational rays, simple curves that are disjoint from K_c , and simple curves in the closures of bounded attracting Fatou components (i.e. Fatou components of K_c).

A sequence x_n in the dynamical plane of g_a or of f_c tends to x_∞ *combinatorially* if for every combinatorial disc D such that $x_\infty \in \text{int}(D)$ all but finitely many x_n are in D . A point x in the dynamical plane of f_c or of $g_{m(c)}$ is *renormalizable* if it belongs to a small connected filled in Julia set; x is *infinitely renormalizable* if it belongs to infinitely many small connected filled in Julia sets.

It follows from Theorem 5.1 that a puzzle piece Y exists for f_c if and only if Y^B exists for $g_{m(c)}$. Consider a copy $\mathcal{M}_{\mathcal{H}} \ni c$ bounded by \mathcal{L}_B from both sides. Then $m(c) \in \mathcal{M}_{\mathcal{H}}^B$. Denote by $\{K_i\}_{i \in I}$ the grand orbit of small filled in Julia sets of f_c associated with $\mathcal{M}_{\mathcal{H}}$; the map $f : I \rightarrow I$ is defined so that $f(K_i) = K_{f(i)}$. There is a unique periodic cycle of $f : I \rightarrow I$; all other filled in Julia sets are preimages of that in the cycle. For $g_{m(c)}$ we have the associated (\mathcal{V}_2 -twin) grand orbit $\{K_i^B\}_{i \in I}$ of small filled in Julia sets such that K_i^B is in Y^B if and only if K_i is in Y for any puzzle piece Y .

We define the *filled in Julia set* $K_{m(c)}$ of $g_{m(c)}$ to be the complement of the attracting basin of the ∞ -0 cycle. The following proposition is the dynamical version of Theorems 6.8 and 2.6.

Proposition 7.3. *Let $\mathcal{M}_{\mathcal{H}} \subset \mathcal{M} \setminus \mathcal{W}_B$ be a copy of the Mandelbrot set bounded by \mathcal{L}_B from both sides and assume that $c \in \mathcal{M}_{\mathcal{H}''} \subsetneq \mathcal{M}_{\mathcal{H}'} \subsetneq \mathcal{M}_{\mathcal{H}}$. Denote by $K_i, K_i^B, K'_j, K_j'^B$ the grand orbits of small filled in Julia sets associated with $\mathcal{M}_{\mathcal{H}}, \mathcal{M}_{\mathcal{H}}^B, \mathcal{M}_{\mathcal{H}'}, \mathcal{M}_{\mathcal{H}'}^B$ and parametrized by $i \in I$ and $j \in J$. Then the following holds in the dynamical planes of f_c and $g_{m(c)}$.*

- *The filled in Julia set $K_{m(c)}$ is locally connected.*
- *If $x_\infty \notin \cup_{j \in J} K'_j$, then a sequence x_n tends to x_∞ combinatorially if and only if x_n tends to x_∞ in the usual topology. If, in addition, $x_\infty \in J_c$, then the intersection of puzzle pieces containing x_∞ in the interior is $\{x_\infty\}$.*
- *If $x_\infty \notin \cup_{j \in J} K_j'^B$, then a sequence x_n tends to x_∞ combinatorially if and only if x_n tends to x_∞ in the usual topology. If, in addition, $x_\infty \in J_{m(c)}^B$ and x_∞ is not a \mathfrak{B} point, then the intersection of puzzle pieces containing x_∞ in the interior is $\{x_\infty\}$.*
- *Let $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$ be the first renormalization map. Then the diameter of preimages of $\overline{W_{\mathcal{H}}^0} \setminus Y_{\mathcal{H}}^p$ under $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$ tends to 0.*
- *Let $g_{m(c)}^p : Y_{\mathcal{H}}^B \rightarrow W_{\mathcal{H}}^B$ be the first renormalization map. Then the diameter of preimages of $\overline{W_{\mathcal{H}}^B} \setminus Y_{\mathcal{H}}^B$ under $g_{m(c)}^p : Y_{\mathcal{H}}^B \rightarrow W_{\mathcal{H}}^B$ tends to 0.*

Proof. We will prove the first, the third, and the fifth claims; the second and the fourth claims are proved in the same way as the third and the fifth claims.

Let

$$K_1''^B, K_2''^B, \dots, K_q''^B$$

be the periodic cycle of small filled-in Julia sets associated with $\mathcal{M}_{\mathcal{H}''}^B$. Then

$$g_{m(c)} : \widehat{\mathbb{C}} \setminus g_{m(c)}^{-1} \left(\{0\} \cup \{\infty\} \bigcup_{i=1}^q K_i''^B \right) \longrightarrow \widehat{\mathbb{C}} \setminus \left(\{0\} \cup \{\infty\} \bigcup_{i=1}^q K_i''^B \right)$$

is a covering of hyperbolic surfaces, and

$$\widehat{\mathbb{C}} \setminus g_{m(c)}^{-1} \left(\{0\} \cup \{\infty\} \bigcup_{i=1}^q K_i''^B \right) \hookrightarrow \widehat{\mathbb{C}} \setminus \left(\{0\} \cup \{\infty\} \bigcup_{i=1}^q K_i''^B \right)$$

is a contracting inclusion.

Recall that the boundaries of components in $\widehat{\mathbb{C}} \setminus K_{m(c)}^B$ are locally connected (Proposition 4.1). By the hyperbolic contraction the diameter of components in $\widehat{\mathbb{C}} \setminus K_{m(c)}^B$ tends to 0. Therefore, $K_{m(c)}^B$ is locally connected. Similarly, the fifth claim is verified.

Let us prove the third claim. The statement is obvious if $x_\infty \notin J_{m(c)}^B$. Assume $x_\infty \in J_{m(c)}^B$ and x_∞ is not a \mathfrak{B} point. If $x_\infty \notin \cup_{j \in J} K_i'^B$, then there are infinitely many m_1, m_2, \dots such that $g_{m(c)}^{m_i}(x)$ is in the interior of $\overline{W_{\mathcal{H}'}^B \setminus Y_{\mathcal{H}'}^B}$. Define $Y_i^B \ni x_\infty$ to be the pullback of $\overline{W_{\mathcal{H}} \setminus Y_{\mathcal{H}}}$ under $g_{m(c)}^{m_i}$ along the orbit of x_∞ . By the hyperbolic contraction the diameter of Y_i^B tends to 0.

If x_∞ is a \mathfrak{B} point, then there are two sequences $Y_i^B, Y_i'^B$ of puzzle pieces constructed as above and two bubbles E, E' such that the diameters of $Y_i^B, Y_i'^B$ tends to 0 and x_∞ is in the interior of $E \cup E' \cup Y_i^B \cup Y_i'^B$ for all i . \square

As a corollary of Proposition 7.3 and 7.1 we have:

Corollary 7.4. *The boundary of a component of $\widehat{\mathbb{C}} \setminus \widehat{K}_c$ or of $\mathbb{S}^2 \setminus K_{m(c)}'^B$ is locally connected. The sets $\widehat{K}_c, K_{m(c)}'^B$ are locally connected.*

All spaces $\mathbb{S}^2 = \mathbb{S}^2(L_B^c)$ are homeomorphic by homeomorphisms preserving $K_{m(c)}'^B$

Proof. The proof is the same as the proof of Proposition 3.8. \square

By definition, the restriction of the equivalence relation L_B^c to K_c is invariant under f_c . We define $g'_{m(c)} : K_{m(c)}'^B \rightarrow K_{m(c)}'^B$ to be the quotient dynamics

$$f_c/L_B^c : K_c/L_B^c \rightarrow K_c/L_B^c.$$

We extend $g'_{m(c)}$ to a branched cover over \mathbb{S}^2 such that $g'_{m(c)}$ has one superattractive periodic cycle with period 2 that attracts all points in $\mathbb{S}^2 \setminus K_{m(c)}'^B$, and locally at the superattractive periodic cycle $g'_{m(c)}$ is conjugate to $z \rightarrow z^2$. We will leave it as an exercise to show that this extension exists (compare to Proposition 2.10).

7.3. Proof of Theorem 1.4. Consider a parameter $c \in \mathcal{M} \setminus \mathcal{W}_B$ such that c is not on the boundary of a hyperbolic component or c satisfies the assumption of Proposition 7.3.

The postcritically finite case of Theorem 1.4 is well known, see [Re], [Tan]. The non-renormalizable non-hyperbolic case of Theorem 1.4 is in [AY]. If c is finitely many times renormalizable and is not hyperbolic, then by Theorem 6.8 triviality of the fiber holds at $m(c)$; the same arguments as in [AY] proves Theorem 1.4 in the non-infinitely renormalizable non-hyperbolic case.

Proposition 7.5. *Under the assumption of Proposition 7.3 the maps $g_{m(c)} : K_{m(c)}^B \rightarrow K_{m(c)}^B$ and $g'_{m(c)} : K_{m(c)}'^B \rightarrow K_{m(c)}'^B$ are topologically conjugate.*

The conjugacy is unique if it preserves the combinatorics of puzzle pieces and coincides with the straightening maps on small filled in Julia sets

Proof. The proof will follow the proof of Theorem 1.2.

We will use the same notations as in Proposition 7.3. Define a bijection $\omega : K'_{m(c)} \rightarrow K^B_{m(c)}$ as follows.

For $\pi(K_i)$, where $i \in I$, define $\omega : \pi(K_i) \rightarrow K^B_i$ to be the dynamical straightening map.

If $x \in K_c$ is the landing point a \mathfrak{B} ray R^ϕ , then denote by x' the landing point of R^ϕ in the dynamical plane of f_B and define $\omega(\pi(x))$ to be $h_a^{-1}(\rho^{-1}(x'))$.

If $x \notin \cup_{i \in I} K_i$ and x is not the landing point of a \mathfrak{B} ray R^ϕ , then define $\omega(\pi(x))$ to be a unique point in $\cap_{Y \ni x} Y^B$, where the intersection is taken over all puzzle pieces containing x .

All above cases are compatible; this defines the bijection $\omega : K'_{m(c)} \rightarrow K_{m(c)}$. Furthermore, ω preserves the combinatorics:

$$(16) \quad \omega(\pi(Y_t \cap K_c)) = Y_t^B \cap K_{m(c)}$$

for a puzzle piece Y_t in the dynamical plane of f_c .

It is also clear that ω is a conjugacy between $g_{m(c)} : K^B_{m(c)} \rightarrow K^B_{m(c)}$ and $g'_{m(c)} : K'^B_{m(c)} \rightarrow K'^B_{m(c)}$. Indeed, by (16) the map ω conjugates the dynamics of puzzle pieces, thus the non renormalizable case follows from Proposition 7.3. On $\cup_{i \in I} K_i$ the map ω coincides with the dynamical straightening map which is, in particular, a topological conjugacy.

We need to show that ω is continuous: if $x_n \in K'_{m(c)}$ tends to $x_\infty \in K'_{m(c)}$, then $\omega(x_n)$ tends to $\omega(x_\infty)$. Consider three cases.

Case 1. Assume $x_\infty \notin \cup_{j \in J} \pi(K'_j)$, then $\omega(x_\infty) \notin \cup_{j \in J} \pi(K'^B_j)$. By (16) the sequence $\omega(x_n)$ tends to $\omega(x_\infty)$ combinatorially. By Proposition 7.3 the sequence $\omega(c_n)$ tends to $\omega(c_\infty)$ in the usual topology.

Case 2. Assume x_∞ and all x_n belong to $\pi(K_i)$ for $i \in I$. Then $\omega(x_n) \in K^B_i$ and the statement follows from the definition of ω because $\omega \circ \pi$ coincides with the dynamical straightening map from K_i to K^B_i .

Case 3. Assume $x_\infty \in \pi(K'_j) \subset \pi(K_i)$ for $i \in I$ and $j \in J$; assume also that all x_n are not in $\pi(K_i)$. Let $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$ be the first renormalization map and let K_1 the periodic filled in Julia set in $Y_{\mathcal{H}}^p$. Define T to be the set of preimages of $\overline{W_{\mathcal{H}}^0} \setminus Y_{\mathcal{H}}^p$ under $f_c^p : Y_{\mathcal{H}}^p \rightarrow W_{\mathcal{H}}^0$; i.e. T is the dynamic decoration tiling. By Proposition 7.3 the diameter of pieces in T tends to 0.

Recall that K_i is a preimage of K_1 ; let n be the minimal number such that $f_c^n(K_i) = K_1$. Define $W(i)$, resp. $T(i)$, to be the preimage of $W_{\mathcal{H}}^0$, resp. of pieces in T , under f_c^n along the orbit of K_i . By shifting the sequence x_n we may assume that all x_n are in $\pi(W(i))$.

Recall that $\pi^{-1}(x_n)$ is either a single point or a pair of \mathfrak{B} points; denote by T_n a piece in $T(i)$ intersecting $\pi^{-1}(x_n)$ and by a_n the point in the intersection of T_n and K_i . As the diameter of pieces in $T(i)$ tends to 0 (because this is so for pieces in T) the distance between $\pi^{-1}(x_n)$ and a_n tends to 0. By Proposition 7.1 the diameter of $\pi^{-1}(x_n)$ tends

to 0. We conclude that a_n tends to $\pi^{-1}(x_\infty)$. By Case 2 the sequence $\omega(\pi(a_n))$ tends to $\omega(x_\infty)$.

Similar to $W(i)$ and $T(i)$ we define $W^B(i)$ and $T^B(i)$ in the dynamical plane of $g_{m(c)}$. It follows from Proposition 7.3 that the diameter of pieces in $T^B(i)$ tends to 0. Therefore, the distance between $\omega(\pi(a_n))$ and $\omega(x_n)$ tends to 0. This implies that $\omega(x_n)$ tends to $\omega(x_\infty)$. \square

Recall that $g'_{m(c)} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is defined so that it has a unique super-attractive two cycle, call it a - b -cycle, the dynamics of $g'^2_{m(c)}$ is locally conjugate to $z \rightarrow z^2$ at a and at b , and all points in $\mathbb{S}^2 \setminus K'_{m(c)}$ are attracted by the a - b -cycle. Therefore, we can extend ω to $K'_{m(c)} \cup V_a \cup V_b$, where V_a, V_b are small neighborhoods of a, b respectively. As

$$g'_{m(c)} : (\mathbb{S}^2 \setminus (K'_{m(c)} \cup g'^{-1}_{m(c)}(\{a\} \cup \{b\}))) \rightarrow (\mathbb{S}^2 \setminus (K'_{m(c)} \cup \{a\} \cup \{b\}))$$

is a covering we can extend ω to $\mathbb{S}^2 \setminus K'_{m(c)}$ such that

$$\omega \circ g'_{m(c)} = g'_{m(c)} \circ \omega.$$

This gives a conjugacy ω between $g'_{m(c)}$ and $g_{m(c)}$ such that the restriction of ω to $K'_{m(c)}$ or to $\mathbb{S}^2 \setminus K'_{m(c)}$ is a homeomorphism. We will leave it as an exercise to show that ω is a homeomorphism on \mathbb{S}^2 (follows from local connectivity of $\mathbb{S}^2 \setminus K'_{m(c)}$).

Let us now argue that if K_c is locally connected, then $f_c \cup_{\mathbb{S}^1} L_B$ is the same as $f_c \cup_{\mathbb{S}^1} f_B$. Define $\Omega_{m(c)} = \mathbb{S}^2 \setminus K'^B_{m(c)}$. By construction, $\Omega_{m(c)}$ is homeomorphic, say by h , to $\text{int}(K_B)$. We need to show that h extends to a homeomorphism between $\overline{\Omega_{m(c)}}$ and K_B . Let E_1, E_2, E_3, \dots be an infinite sequence of components in $\text{int}(K_B)$ such that the intersection $\overline{E_i} \cap \overline{E_{i+1}}$ consists of one point; in particular, $E_i \neq E_j$ if $i \neq j$. We claim that h extends to a homeomorphism from $\overline{\cup E_i}$ onto $\overline{\cup h(E_i)}$. Indeed, it follows from the local connectivity of E_i that h extends to ∂E_i for every i . Therefore, it is sufficient to show that the sequence E_i accumulates at one point. If E_i were accumulating at two points, say x, y , then the sequence $\pi^{-1}(E_i)$ of components in $\widehat{\mathbb{C}} \setminus \widehat{K_c}$ would accumulate at $\pi^{-1}(x)$ and $\pi^{-1}(y)$ which is the contradiction to the local connectivity of K_c ; details are left to the reader.

REFERENCES

- [AY] M. Aspenberg, M. Yampolsky, *Mating non-renormalizable quadratic polynomials*, Comm. Math. Phys. 287 (2009).
- [DH1] A. Douady, J.H. Hubbard, *On the dynamics of polynomial-like maps*. Ann. Sci. Ecole Norm. Sup., Vol. 18 (1985), 287–343.
- [DH2] A. Douady, J.H. Hubbard. *Étude dynamique des polynômes complexes*. Publications Math. d'Orsay, 84-02 (1984) and 85-04 (1985).
- [DH3] A. Douady, J.H. Hubbard. *A proof of Thurston's topological characterization of rational functions*. Acta Math. V 171 (1993) 263–297.
- [Du] D. Dudko, *The decoration theorem for Mandelbrot and Multibrot sets*. Preprint. ArXiv 2010.

- [HY] J. G. Hocking, G. S. Young, *Topology*, Sec. edit. Dover Public, Inc., New York, 1988.
- [Hu] J.H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*. In: *Topological Methods in Modern Mathematics*, 467–511 and 375–378 (figures), Publish or Perish, (1993).
- [Lu] J. Luo, *Combinatorics and Holomorphic Dynamics: Captures, Matings and Newton's Method*, PhD Thesis, Cornell University, 1995.
- [Ly1] M. Lyubich, *Dynamics of quadratic polynomials, I-II*. *Acta Math.* Vol. 178 (1997), 185–297.
- [Ly2] M. Lyubich, *Feigenbaum-Collet-Tresser Universality and Milnor's Hairiness Conjecture*. *Ann. Math.*, Vol. 156 (1999), 1–78.
- [Mc1] C. McMullen, *Complex dynamics and renormalization*. *Ann. of Math. Stud.*, Princeton Univ. (1994).
- [Mc2] C. McMullen, McMullen. *The Mandelbrot set is universal. The Mandelbrot set, theme and variations*. *Cambridge University Press* (2000), 1–17.
- [Mi1] J. Milnor, *Periodic orbits, external rays and the Mandelbrot set: an expository account*. In: *Géométrie complexe et systèmes dynamiques*, *Astérisque* Vol. 261 (2000), 277–333.
- [Mi2] J. Milnor *Local connectivity of Julia sets: Expository lectures*. Preprint (1992), IMS at Stony Brook.
- [Mu] R. L. Moore, *Concerning upper-semicontinuous collections of continua*, *Transactions of the AMS*, 27, Vol. 4 (1925), 416–428.
- [Ne] V. Nekrashevych. *Self-similar groups*, volume 117 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., Providence, RI, 2005.
- [Tan] Tan, L. *Matings of quadratic polynomials*. *Erg. Th. and Dyn. Sys.* V 12 (1992), 589–620.
- [Ti1] V. Timorin. *External boundary of M_2* . Fields Institute Communications Volume 53: Holomorphic Dynamics and Renormalization. A Volume in Honour of John Milnor's 75th Birthday.
- [Ti2] V. Timorin. *Parameter slices of quadratic rational functions*. Webpage <http://math.jacobs-university.de/timorin/perk/perk.html>
- [PR] C. L. Petersen, P. Roesch, *Carrots for dessert*. Preprint. ArXiv 2010.
- [Re] M. Rees. *Realization of matings of polynomials as rational maps of degree two*. Manuscript, 1986.
- [Sc1] D. Schleicher. *On fibers and local connectivity of Mandelbrot and Multibrot sets*. *Fractal geometry and applications: a jubilee of Benoît Mandelbrot. Proceedings of Symposia in Pure Mathematics*. V 72 (2004), 477–517.
- [Sc2] D. Schleicher, *Symbolic dynamics of quadratic polynomials*. Manuscript in preparation.
- [Wh] G. Whyburn, *Analytic Topology*, AMS Colloq. Publ. 28, 1942.
- [Wi] B. Wittner, *On the bifurcation loci of rational maps of degree two*, PhD Thesis, Cornell University, 1988.